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The Automatic Control of an Aeroplane in the Landing Approach onto an Aircraft Carrier

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Summary: An investigation is made into the possibility of the completely automatic longitudinal control of an aircraft in the deck landing approach. The method presupposes the existence of a radio beam system transmitted from the carrier and a study is made of the motion of the aircraft down this beam under automatic control.

The longitudinal motion alone of the aircraft and carrier is considered. The motion of the carrier is assumed to consist of a steady forward velocity modified by a non-pitching sinusoidal oscillation in the vertical plane.

The most successful method of approach control proposed is based on the use of a single beam emanating from a source at a fixed distance above the carrier deck and fixed in direction. The preliminary pilot-controlled approach is assumed to be down the mean position of the beam, and the auto-control requires, over and above certain ideal variations of elevator angle and thrust, additional types of control of the following forms. On the elevator, displacement pitch and displacement track error control; and on the thrust, displacement track error control.

Information is also forthcoming regarding the unsuitability of certain other types of control which at first sight seem promising.

1. Introduction

In recent years the blind landing of land-based aircraft has been given a great deal of attention and several systems of approach have been designed. Pritchard[1] has given a general account of various existing types of blind approach all of which involve the use of radio beams and radar. The fundamental idea behind each different system is essentially the same, namely, to keep the aircraft on some predetermined flight path down onto the runway. To this end information regarding the position of the aircraft relative to the approach beams is periodically given to the pilot so that he may make certain adjustments to the controls and so centralise the aircraft in the beam system.

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The purpose of the present paper is to investigate the possibility of a similar technique to that just described being employed in the deck landing approach. In this case, however, the landing area is very limited in extent and, in general, is not stationary, so that a high degree of accuracy is required in the response of the pilot to the "error" stimuli supplied to him. Moreover, in bad weather the resources of the pilot will already be severely taxed and therefore automatic control is desirable to relieve him of the responsibility of guiding the machine down the beam. Furthermore, the response of an auto-pilot depends solely upon the input signals and thus can be assessed once and for all. From all viewpoints, therefore, it is to be expected that a more satisfactory approach will ensue if the "error" radio-signals are fed directly to the auto-pilot. The problem thus resolves into an investigation into the automatic control of the deck landing approach, and in this paper its longitudinal aspects are considered.

The discussion separates into two distinct parts. Part I deals with the establishment of a beam system, and the general design of an auto-pilot suitable for the approach. Part II combines the general conclusions of the previous part in a detailed analysis of a Single Beam Approach system.

Notation

A	initial aircraft distance from beam source
a	amplitude of carrier oscillation
c	$=[an/(V_e\hat{t})]\cos\phi$ (*see* equation (1) of Appendix I)
C_{D_0}, C_{Df}	profile drag coefficient, flaps up and flaps down respectively
C_{Di}	induced drag coefficient
C_D, C_L	drag and lift coefficients of complete aircraft
C_{LT}	lift coefficient of tail unit
$\bar{c}$	mean wing chord
e	as suffix, denotes equilibrium conditions
H	aircraft height above mean deck level
i	as suffix, denotes ideal conditions
i_B	coefficient of inertia about y-axis (*see* Ref. 4)
k, l	ideal incidence parameters (*see* equation (6) of Appendix I)
l_T	tail arm
k_e	$=\frac{1}{2}C_L$
N, n	dimensional and dimensionless circular frequencies of carrier oscillation
S, S'	wing and tailplane areas

T	thrust in lb.
t	time in seconds
t_1, τ_1	constants defining carrier displacement at auto-pilot engagement
$\hat{t}$	unit of aerodynamic time, the aerosecond
U	velocity along x-axis
u	increment of U from its equilibrium value
V_c	carrier forward velocity relative to sea
V_e	steady aircraft velocity in preliminary approach
$\hat{V}$	$=(V_c/V_e)\cos\phi$ (*see* equation (1) of Appendix I)
W	velocity along z-axis
W_1	aircraft weight
w	increment of W from its equilibrium value
x_T	is given by:—thrust increment in lb. $=\rho S V_e^2 x_T$
z^*	is given by:—track error in ft. $=V_e\hat{t}z^*$
α, α'	wing and tailplane incidences
β	ideal flight angle down beam (*see* equation (1) of Appendix I)
β_m	flight angle down mean line of beam (*see* equation (2) of Appendix I)
γ	climbing angle of x-axis
ε	angle of downwash at tail
ϕ	beam angle of elevation
η	increment in elevator angle
θ	increment in attitude
λ	aspect ratio
μ	relative aircraft density
ρ	air density
τ	aerodynamic time $(t=\hat{t}\tau)$

PART I

GENERAL DESIGN OF THE AUTO-PILOT AND BEAM LAYOUT

2. Establishment of a Beam System

2.1. General requirements

First of all it is necessary to say a little about the motion of the aircraft carrier itself. This will generally consist of a number of oscillations about a steady state in which the ship steams forward on an even keel. Here all movement not confined

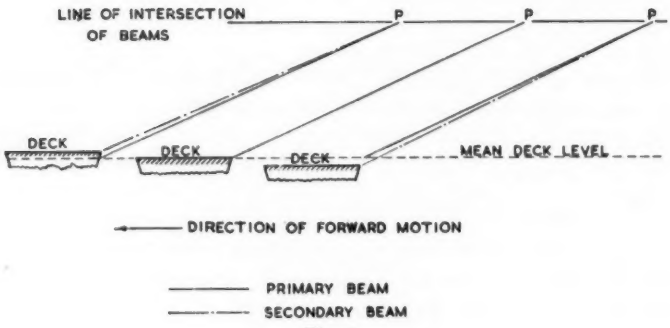


Fig. 1.

to the longitudinal plane is neglected, and in that plane it is assumed that the carrier steams steadily ahead and possesses, in addition, a non-pitching sinusoidal oscillation in the vertical plane. Pitching as such is unimportant since only the landing *approach* is being considered and not the touch down, so that its effect on the attitude of the aircraft relative to the deck is of no weight and its effect on beam movement may be removed by suitable stabilisation.

Now, whatever be the beam system established, the following general requirements should be satisfied. First, the early stages of the approach should be a steady, straight-line flight with no attempt made to follow the actual oscillations of the carrier, and secondly, the ultimate flight path must be towards the touch-down point on the deck. The first stage may be under either human or automatic control; the second must be fully automatic. It is necessary to make sure that the ideal flight, in which the aircraft keeps on the beam, is practicable in the sense that excessive elevator movements and thrust changes or stalling do not occur. It is then further necessary to determine whether the auto-controlled flight remains practicable and stable as the machine attempts to stay on the beam in the presence of gusts and other disturbing phenomena.

It might be added that throughout the complete approach the aircraft should be trimmed for landing conditions, namely, flaps and undercarriage down, and so on.

2.2. *Twin beam system*

The fact that a distinction is made between the early and final stages of the approach suggests that the possibilities of a twin beam system should be investigated. One beam, the primary, say, should be stabilised relative to the oscillation of the carrier with its direction in space fixed, although still possessing the forward motion of the ship; this will provide a straight-line flight path for the preliminary approach. The other, a secondary beam, should have its source localised in the carrier deck, in the same vertical as that of the primary, with its direction constrained to intersect

the primary at a constant distance, say, from the stabilised source of the latter. Three successive positions of the carrier and the beam system are depicted in Fig. 1. In the approach the pilot should maintain steady, straight-line flight down the primary beam and at some point of this path engage the auto-pilot. This will effect the change-over to the secondary beam on arrival at P and will then control the flight down this beam onto the deck.

It turns out, however, that the Twin Beam Approach system must be rejected, as it is found that the ideal flight down the secondary beam is totally impracticable owing to the demand for a large negative thrust in following the seemingly increasing oscillations of the deck, as viewed from the aircraft.

2.3. *Single beam system*

The detailed study of the twin beam system leads to the conclusion that it becomes more and more satisfactory as the point of intersection of the two beams recedes farther away. In the limit this point is allowed to go to infinity, whereupon the two beams become parallel and the primary one may be dispensed with. There remains a single beam emanating from a source in the deck*, stabilised at a given inclination to the horizontal, and moving bodily up and down with the motion of the carrier. Such a system is fully discussed in Section 6 and an outline of the theoretical analysis is given in Appendix I. It will be shown that this type of blind-approach layout appears to be satisfactory in all essential details.

3. Automatic Longitudinal Control about Unsteady States of Motion

The automatic control of aircraft is now common practice and considerable literature exists on the subject. An article by Meredith and Cooke[2] discusses the general problems of auto-control and a review of the methods employed is given by Weiss[3]. More recent papers by Neumark[4] and Sudworth and Hopkin[5], [6] considerably clarify the theoretical treatment†. However, several important differences exist between the control equations developed by the foregoing and those required for the present work. Accordingly, the general auto-control formulae in any longitudinal motion will be derived with special reference to these differences.

Analytically the longitudinal motion of an aeroplane is referred to a certain, steady, straight-line flight, known as Equilibrium Flight, in which an orthogonal system of wind axes is established. The small deviations of velocity, incidence, attitude and so on from their equilibrium values are known as increments. In the

*As remarked in Section 8, the beam source does not actually need to be in the deck but, taken there, it simplifies the analysis considerably.

†Since this work was concluded, Meredith[8] has given a survey of modern trends in auto-pilot design, and Taylor[9] has discussed a generalised theory of aircraft auto-control.

general motion one may specify that these increments should take up prescribed values related to corresponding settings of the controls. Thus, in the case of prescribed, steady, straight-line flight, the several values are taken to be zero; more generally they may be *prescribed functions of the time*. The purpose of auto-control is then to suppress any variations of the flight increments from these ideal values.

In what follows the dimensionless notation of Neumark[4] is employed throughout. Ideal values of the increments, together with their time rates of change, are specified by the suffix "i," *e.g.* the attitude increment is θ_i. The actual values of the increments which occur in flight are written without the suffixes, *e.g.* θ. Error terms arise, therefore, of the type $(\theta-\theta_i)$. The control is made automatic by gearing the control movements to linear combinations of these error terms. Thus, typical formulae of automatic elevator and automatic thrust control become

$$\left.\begin{aligned}\eta=\eta_i+f_\theta(\theta-\theta_i)+f_{d\theta/d\tau}\left(\frac{d\theta}{d\tau}-\frac{d\theta_i}{d\tau}\right)+f_u\left(\frac{u}{U_e}-\frac{u_i}{U_e}\right)+f_{du/d\tau}\left\{\frac{d}{d\tau}\left(\frac{u}{U_e}\right)-\frac{d}{d\tau}\left(\frac{u_i}{U_e}\right)\right\}\\ +f_w\left(\frac{w}{U_e}-\frac{w_i}{U_e}\right)+f_{dw/d\tau}\left\{\frac{d}{d\tau}\left(\frac{w}{U_e}\right)-\frac{d}{d\tau}\left(\frac{w_i}{U_e}\right)\right\}\\ x_T=x_{Ti}+e_\theta(\theta-\theta_i)+e_{d\theta/d\tau}\left(\frac{d\theta}{d\tau}-\frac{d\theta_i}{d\tau}\right)+e_u\left(\frac{u}{U_e}-\frac{u_i}{U_e}\right)+\quad .\quad .\quad .\quad .\end{aligned}\right\}\quad(1)$$

where f_θ, $f_{d\theta/d\tau}$, f_u, $f_{du/d\tau}$, f_w, $f_{dw/d\tau}$ are constant elevator control gearings and e_θ, $e_{d\theta/d\tau}$, e_u, $e_{du/d\tau}$, e_w, $e_{dw/d\tau}$ are constant thrust control gearings.

Automatic controls such as those devised would give the required flight with regard to speed and attitude but would neither be able to localise the line of flight in space nor to maintain it along a beam. Such a system is useless here and a form of control needs adding to force the aircraft to ride the beam. Hopkin[7] has suggested a way of doing this in the case of lateral motion, introducing the concept of track error, and the method is now applied to the longitudinal case.

The track error, z^*, is defined as the perpendicular distance of the aircraft centre of gravity from some required flight path. The sign is determined by taking z^* positive in the general direction of the positive z-axis. Thus in level flight a positive track error means that the flight is below the required path. In the general case a kinematic expression can be derived relating $dz^*/d\tau$ to the flight increments.

Automatic track error control is obtained by gearing the elevator and thrust directly to the variations of z^* in the form

$$\left.\begin{aligned}\eta&=-f_{dz/d\tau}\frac{dz^*}{d\tau}-f_z z^*-f_{\bar{z}}\int z^*d\tau\\ x_T&=e_{dz/d\tau}\frac{dz^*}{d\tau}+e_z z^*+e_{\bar{z}}\int z^*d\tau\end{aligned}\right\}\quad .\quad .\quad .\quad .\quad(2)$$

where $e_{dz/dt}$, e_z, $e_{\bar{z}}$, $f_{dz/d\tau}$, f_z, $f_{\bar{z}}$ are additional constant thrust and elevator control gearings.

Complete automatic longitudinal control is obtained by combining the track error controls (2) linearly with the flight controls (1). From a practical aspect such auto-control involves the measurement and use of recorded quantities yielding error terms to which the elevator and thrust are related with appropriate "gear ratios." The general relationships between these recorded quantities and the aircraft control settings may thus be written as

$$\left.\begin{aligned}\eta &= \eta_1 + f_\theta(\theta-\theta_1) + \ldots\ldots + f_{dw/dt}\frac{d}{d\tau}\left(\frac{w}{U_e}-\frac{w_1}{U_e}\right) - f_{dz/dt}\frac{dz^*}{d\tau} - f_z z^* - f_{\bar{z}}\int z^* d\tau \\ x_T &= x_{T1} + e_\theta(\theta-\theta_1) + \ldots\ldots + e_{dw/dt}\frac{d}{d\tau}\left(\frac{w}{U_e}-\frac{w_1}{U_e}\right) + e_{dz/dt}\frac{dz^*}{d\tau} + e_z z^* + e_{\bar{z}}\int z^* d\tau\end{aligned}\right\} \quad (3)$$

4. Types of Auto-pilot Suitable for the Landing Approach

The broad features of the design of the auto-pilot must be considered before an approach down a particular beam system is investigated in detail. The question to be answered is—what types of control need to be employed so that the "perturbation" motion, as defined in Appendix I, is stable? Owing to the labour involved in the calculations, only a limited number of the types of control are investigated; those retained may be specified by the gearings.

It is necessary to make a simplified response calculation to obtain some idea of the value of any particular control gearing, and some typical values, together with the criteria involved in the assessment, are found to be

$\delta f_\theta = 22$: 0.5 deg. elevator per deg. attitude error

$\delta f_z = 16$: 1 deg. elevator per 25 ft. track error

$e_z = 0.7$: 3 ft./sec. rate of climb per 20 ft. track error.

4.1. *Some results concerning* ($e_{\bar{z}}$, $f_{\bar{z}}$, f_z, $f_{dz/dt}$) *controls*

It can be shown that in gliding flight at high incidences $f_{\bar{z}}$, f_z, $f_{dz/dt}$ controls are always unstable. The limiting incidence for stability is given where $C_{D0} = C_{Di}$. In powered flight at *low* thrusts the same result holds and the value of the critical incidence is still approximately that for the corresponding glide. With *high* thrusts, however, an entirely different picture may be presented. In fact the extra upward force due to the tilting of the thrust axis may become a dominant factor and may actually lead to stability (as shown later in this section).

It may be shown further that an e_z control leads to oscillatory instability always, save possibly for a very narrow range of values of ω, bounded above and below by regions giving definite instability for all values of the gearing.

4.2. *Stable combinations of control*

With Low Thrust. A satisfactory solution to the problem is obtained by combining an f_θ control with an ordinary e_z type. The f_θ control needs to be used to maintain a satisfactory distribution of the total damping between the various modes of motion, while the e_z control is necessary for following the beam. It may be shown that, with the relevant data of Appendix II, this combination is stable for the following values of the gearings

$$\delta f_\theta = 0, \quad e_z < 1.3 \qquad \delta f_\theta = 20, \quad e_z < 2.3$$
$$\delta f_\theta = 10, \quad e_z < 1.8 \qquad \delta f_\theta = 30, \quad e_z < 2.6.$$

It is easy to show that a control strength $\delta f_\theta = 20$ is a suitable one to employ, and in this case the stability roots for varying e_z may be tabulated as

e_z Control	Subsidence	Slow Mode	Rapid Mode
0	—	$-1.5850 \pm 0.4591i$	$-1.9776 \pm 3.8290i$
0.5	-2.2059	$-0.4756 \pm 0.6026i$	$-1.9840 \pm 3.8316i$
0.75	-2.3714	$-0.3898 \pm 0.8181i$	$-1.9871 \pm 3.8330i$
1.0	-2.4990	$-0.3230 \pm 0.9661i$	$-1.9901 \pm 3.8344i$

Besides introducing a rapid subsidence an e_z control has a marked effect on the frequency and damping of the slower mode. Apparently nothing is to be gained from the use of a strong gearing and a suitable combination should prove to be

$$\delta f_\theta = 20, \quad e_z = 0.5 \qquad . \quad . \quad . \quad . \quad . \qquad (4)$$

With High Thrust. The (e_z, f_θ) combination already devised is considered first. It may be shown that, with the data of Appendix II relevant to the present case, gearings $\delta f_\theta = 20$, $e_z = 0.75$ should prove satisfactory. These values, relating to flight with a moderately high mean thrust, lead approximately to the same damping distribution as do those considered suitable for flight at low thrust.

It has also been stated that an f_z control is destabilising, except possibly in this case of flight with high thrust. Such control is now combined with the two foregoing types and, retaining the values of $e_z (= 0.75)$ and $\delta f_\theta (= 20)$, the stability roots are compared when $f_z = 0$ and 0.15.

f_z Control	Subsidence	Slow Mode	Rapid Mode
0	-2.3649	$-0.4426 \pm 0.7958i$	$-1.9825 \pm 3.8184i$
0.15	-1.4619	$-0.8565 \pm 0.8288i$	$-2.0201 \pm 3.6724i$

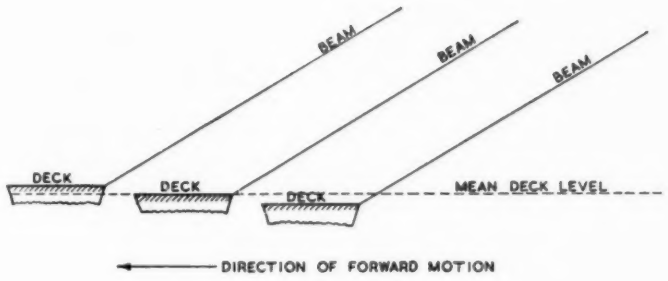


Fig. 2.

It is seen that the small gearing introduced has the important effect of removing damping from the rapid subsidence and transferring it to the slow mode. In view of this result these three controls should be combined in the auto-pilot with the gearings

$$\delta f_\theta = 20, \quad f_z = 0.15, \quad e_z = 0.75 \quad . \quad . \quad . \quad . \quad (5)$$

PART II

A SINGLE BEAM APPROACH SYSTEM

5. General Details

In this part of the paper the single beam layout proposed in Section 2 and the auto-pilot "designed" in Section 4 are combined to give a single beam approach system. The scheme envisaged consists of a single beam, fixed in direction, making a constant angle ϕ with the carrier deck. Its source is localised in the deck, thereby possessing the complete carrier motion. Fig. 2 shows three successive positions of the carrier and beam during the motion.

At some large distance from the carrier the aircraft should be trimmed for the landing approach and, under the control of the pilot, should be guided towards the beam and maintained in flight down the mean line of it. At some stage of this flight the auto-pilot should be engaged and this will then direct the aircraft into the oscillating beam and down the beam, on to the deck. The theoretical analysis of the complete flight with this preliminary approach is outlined in Appendix I.

6. Ideal Flight Down The Beam

The formal expressions for the ideal increments are given in Appendix I as the set of equations (10)-(14). The variations of flight from that down the mean line of the beam are of a sinusoidal nature of the same frequency as the carrier oscillation. Consider the thrust. It is clearly important to maintain a large thrust

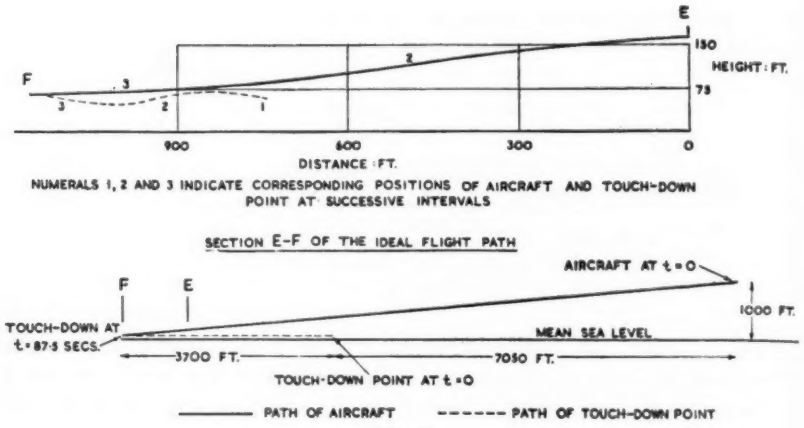


Fig. 3.

in the preliminary approach down the mean line of the beam since the amplitude of the ideal thrust oscillation may be of considerable magnitude, and a large negative value is incapable of being produced by any conventional aircraft. This implies that the beam angle must be small and the mean angle of flight, β_m, must be considerably less than the gliding angle. Such a value is taken in the numerical data of Appendix II.

Further, it may be shown that it is necessary to incorporate a prescribed variation of the incidence in the ideal flight. With a sinusoidally oscillating incidence with parameters k and l as laid down in Appendix I, the amplitudes of the ideal flight increments are themselves functions of k and l. These parameters are chosen so as to specify such incidence changes as cause the smallest corresponding oscillations of speed, attitude and control movements consistent with not stalling the aircraft as it flies down the beam.

Values of k and l which satisfy these requirements are

$$k = -0.06 \quad l = 0.01 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and the ideal flight down the beam takes place with the following oscillation of the type $d\cos(nt+\varepsilon)$ and period 8 sec.

	Amplitude (d)	*Mean Setting*
Incidence: deg.	3.5	12
Speed: ft./sec.	3.9	123.6
Attitude: deg.	5.4	−4.9
Thrust: lb.	370	1090
Elevator angle: deg.	2.2	—

Moreover, defining the normal acceleration as the reversed acceleration along the z-axis, the maximum excess value caused by the manoeuvres is about 0.2 g.

Thus an ideal flight down the beam is obtained, capable of practical achievement by a conventional aircraft. The flight path relative to the touch-down point on the deck is drawn in Fig. 3 and the variations of the flight increments and control settings are plotted in Figs. 4 to 8.

7. Actual Flight Down the Beam with (e_z, f_θ, f_z) Auto-control

Having determined a satisfactory ideal motion for the aeroplane there remains the final design of an automatic control system which will guide the aircraft into the beam and cause its motion actually to approach the ideal. Alternative stable combinations of control have been examined in Section 4 and their application will now be discussed. Here the ideal flight demands a high mean thrust and thus a combination incorporating f_z control may be used and so the beam-riding element may be strengthened. Therefore an auto-pilot combining (e_z, f_θ, f_z) controls with the stable set of gear ratios determined previously (equation (5)) is considered.

The actual flight is now calculated as the sum of the ideal motion and the transient perturbations. The ideal motion has been obtained already and discussed in Section 6. The transient perturbations are calculated as the complete solutions of equations (14) of the Appendix.

Three sets of non-ideal initial conditions at auto-pilot engagement are taken in the assessment of the perturbations, and in all cases the unfavourable initial displacement of the carrier deck from its mean level is assumed to be a maximum.

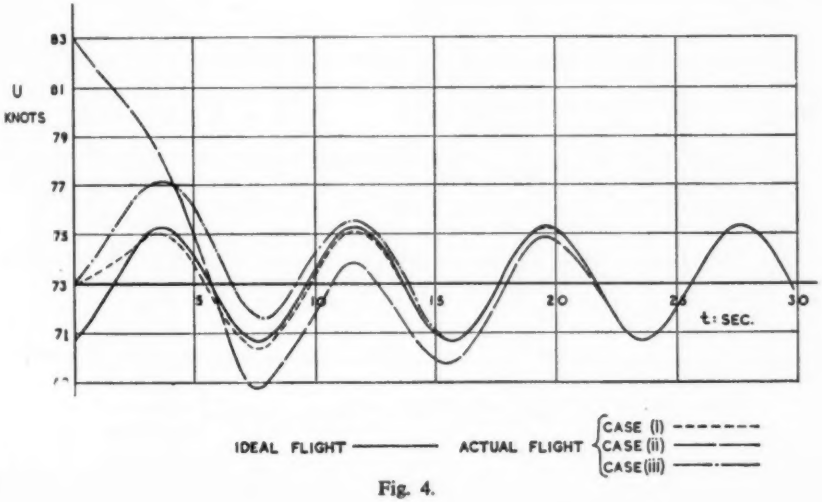


Fig. 4.

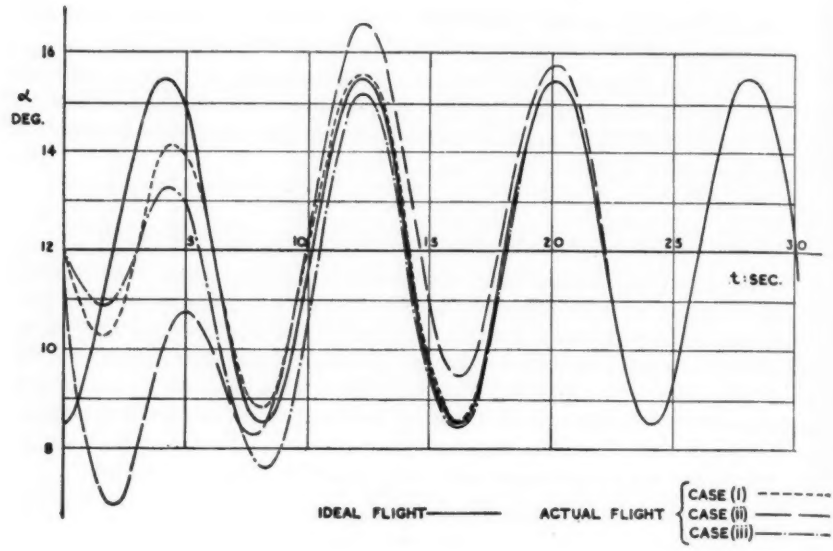
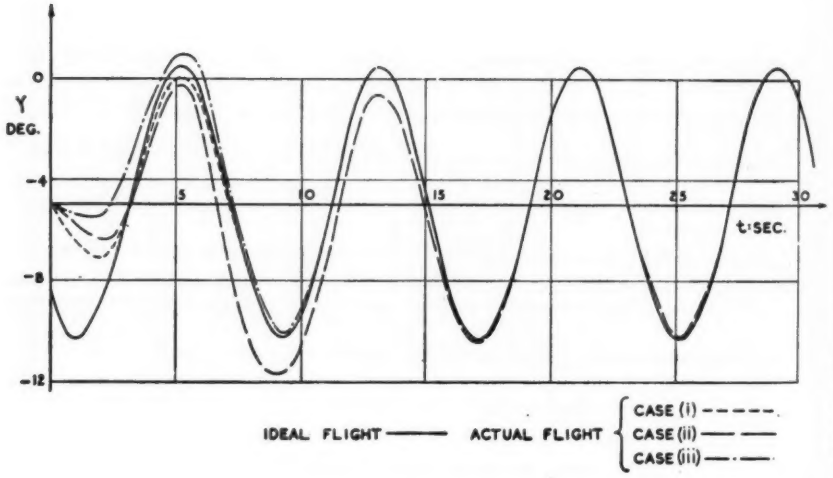

α
DEG.
16
14
12
10
8
5
10
15
20
25
30
t: SEC.
IDEAL FLIGHT
ACTUAL FLIGHT
CASE (i)
CASE (ii)
CASE (iii)

Fig. 5.


γ
DEG.
0
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-8
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5
10
15
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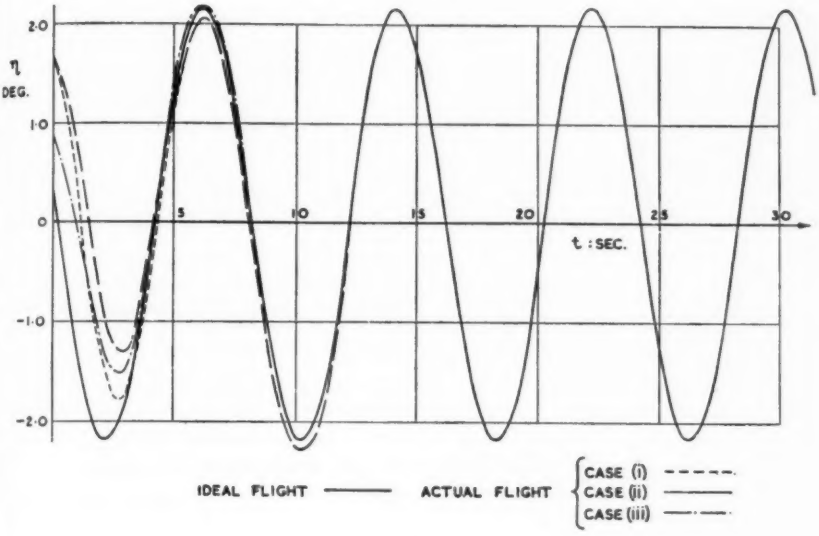
25
30
t: SEC.
IDEAL FLIGHT
ACTUAL FLIGHT
CASE (i)
CASE (ii)
CASE (iii)

Fig. 6.


η
DEG.
2·0
1·0
0
-1·0
-2·0
5
10
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25
30
t : SEC.
IDEAL FLIGHT
ACTUAL FLIGHT
CASE (i)
CASE (ii)
CASE (iii)

Fig. 7.

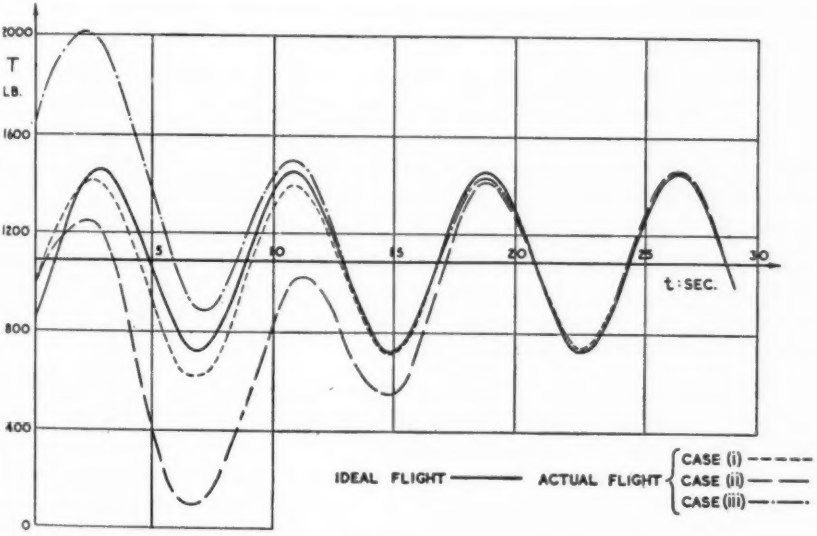
T
LB.
2000
1600
1200
800
400
0
5
10
15
20
25
30
t : SEC.
IDEAL FLIGHT
ACTUAL FLIGHT
CASE (i)
CASE (ii)
CASE (iii)

Fig. 8.

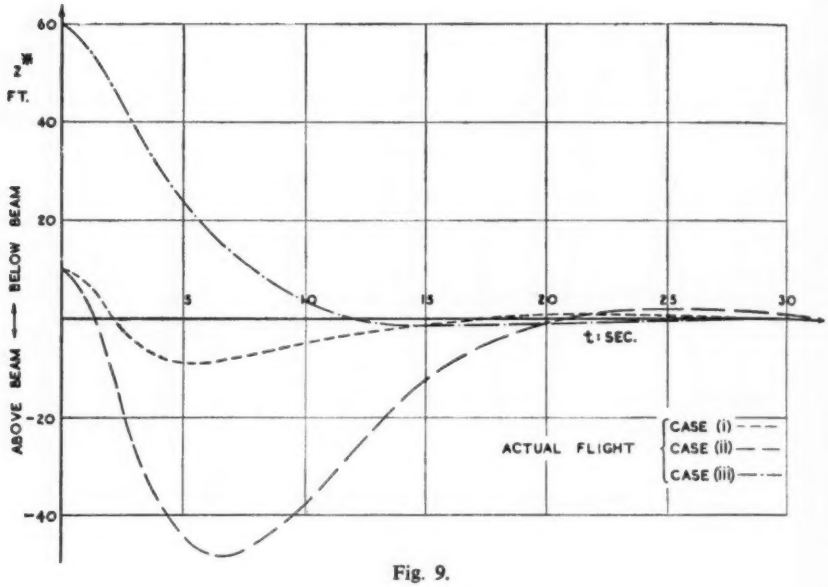

Fig. 9.

Of the cases chosen, the first is the best that could be expected to occur in practice; the remainder introduce serious flight errors at auto-pilot engagement. They are

(*i*) Aircraft in equilibrium flight down the mean line of the beam.

(*ii*) Aircraft in flight down the mean line of the beam with a speed error from equilibrium of 10 knots (actual speed = 83 knots).

(*iii*) Aircraft in equilibrium flight parallel to that down the mean beam with a 50 ft. displacement error (track error = 50 ft. + initial beam displacement).

The results of the calculations of the perturbations, and thus of the actual motion, in each of these cases is plotted in Figs. 4 to 8. The track error is plotted separately in Fig. 9, while Fig. 10 shows the actual flight path of the aircraft down the beam in the ideal case, together with the two worst cases of initial conditions, plotted until the perturbations have disappeared.

A study of the curves leads to the following conclusions. In all cases a satisfactory response ensues, the stall is avoided and ample thrust is available. The total time of flight from auto-pilot engagement at about 1,000 ft. to touch-down on the deck is approximately 87.5 sec., and for the last 55 seconds of this the approach takes place ideally. In practice, conditions at auto-pilot engagement will generally be better than those discussed here and in such cases the approach will certainly be possible.

On an actual aircraft the presence of friction and inertia in the control circuits will remove the "kicks" that appear in a theoretical treatment of displacement type controls. It may be shown that a viscous friction type of control lag does not affect the foregoing conclusions adversely; in fact it merely removes these kicks from the theoretical response.

Finally, as may be expected, it can be shown that the effect of a headwind is to bring about an all-round improvement in the approach.

8. Some Practical Aspects

The paper as a whole presents a description of a possible approach technique, with but a single example worked out, rather than a final design for an auto-controlled approach system. For example, it may be considered that the total time of approach-flight, namely 87.5 sec., is too long for practical purposes. This defect may be remedied by slightly modifying the approach servos so that, for example, the auto-pilot engagement takes place at 500 ft. instead of at 1,000 ft. as in the case worked out. This would result in a total approach time of about 44 sec. with the last 11 sec. ideal.

Again, the auto-pilot gearings as finally given numerically in the paper are certainly not the best suited to the purpose. Taylor[9] has shown that the "best" response to any input signal is produced when the gearings are designed for critical damping—best in the sense of least peak load or overshoot for a given transient decay time—and clearly such a modification would help in the present problem. A possible result would be to make it possible to lower the auto-pilot engagement height still further, thus obtaining a smaller total approach-flight time retaining the same final 11 seconds of idealised approach.

The analysis is based upon the fact that a prescribed variation of incidence in ideal flight down the beam demands a certain corresponding thrust and elevator setting. It may be shown that, for any particular aircraft, these resulting control

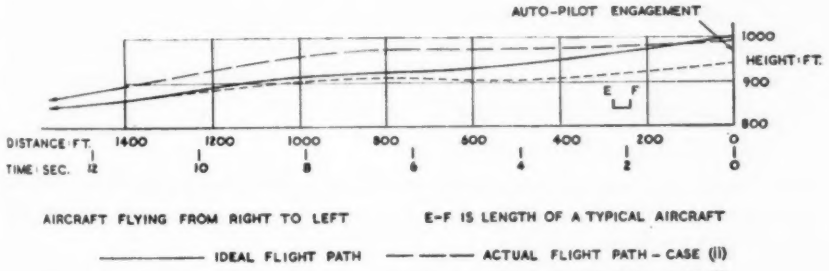
AUTO-PILOT ENGAGEMENT
1000
HEIGHT: FT.
900
800
E F
DISTANCE: FT. 1400 1200 1000 800 600 400 200 0
TIME: SEC. 12 10 8 6 4 2 0
AIRCRAFT FLYING FROM RIGHT TO LEFT
E-F IS LENGTH OF A TYPICAL AIRCRAFT
IDEAL FLIGHT PATH
ACTUAL FLIGHT PATH – CASE (ii)
ACTUAL FLIGHT PATH – CASE (iii)

Fig. 10.

movements may be written purely as functions of the carrier oscillation so long as the violence of this is less than that specified in Appendix II. The basic servo-mechanisms are therefore relatively simple, being merely sinusoidal functions of two variables.

Next, although the beam source has been taken in the deck it is unnecessary to persist in such an awkward location. A suitable position would be at the masthead and this has the added advantage of cutting down "ground return." The track error control would then be so designed as to centre on a line parallel to the guiding beam.

In conclusion it is remarked that although a simple carrier oscillation has been assumed, the approach system is applicable to more general carrier motions provided that these can always be resolved by Fourier Analysis into sinusoidal non-pitching oscillations in the vertical plane.

ACKNOWLEDGMENT

The paper is a summary of a dissertation for the degree of Ph.D. on work done in the Department of Mathematics at the University of Leeds, and the author wishes to acknowledge his indebtedness to Dr. H. L. Price, of that department, for his aid and criticism throughout the work.

APPENDIX I

Theoretical Analysis of a Single Beam Approach System with a Preliminary Approach Down the Mean Line of the Beam

The beam approach system and the approach flight to be analysed are described in Section 5 and the assumed carrier motion is as stated in Section 2 of the main paper. A first approximation is taken throughout the analysis.

Two distinct problems are involved: —

To determine the nature of the motion down the beam perfectly executed.

To find what is the motion possessed by the aircraft as the auto-pilot controls it in an attempt to follow the beam.

The following definitions are assumed: —

Ideal Motion consists of the steady flight down the mean line of the beam and the subsequent perfect motion with a prescribed variation of incidence as the aircraft flies down the oscillating beam.

Actual Motion consists of the auto-pilot engagement and the subsequent flight as the aircraft attempts to follow the oscillating beam.

1. Kinematic Preliminaries

On the assumption that the aircraft c.g. remains on the beam at all times in ideal flight, the angle of descent down the beam may be determined from purely geometrical considerations. Fig. A.1 shows the positions of the beam source (D, D'), beam $(DS, D'S')$ and aircraft (S, S') at two consecutive instants of time, with the vertical carrier displacement at any instant expressed as $a \sin N(t+t_1)$.

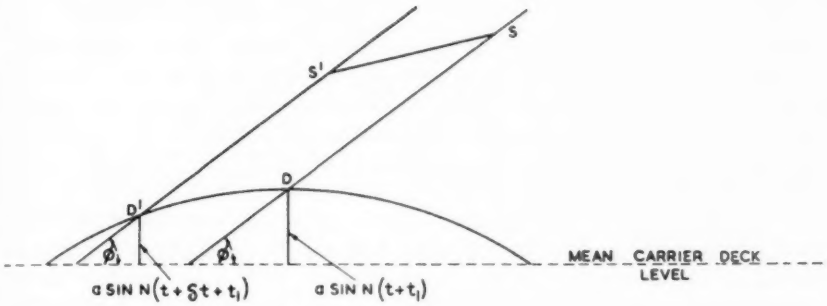

Fig. A.1.

Thus, by simple geometry, if $(V_e \sin\phi)$ and (aN) are small compared with V_e, and ϕ is a small angle, the downward angle of flight along the oscillating beam becomes

$$\beta = \phi - \hat{V} \tan\phi - c \cos n(\tau + \tau_1), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the downward angle of flight along the mean line of the beam is

$$\beta_m = \phi - \hat{V} \tan\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the height above mean carrier deck level may be deduced as

$$H = A \sin\phi - V_e(1-\hat{V}) \sin\phi \,.\, t + a \sin N(t+t_1) \,. \quad . \quad . \quad . \quad (3)$$

2. Axes of Reference

Equilibrium wind axes and velocities are chosen at any convenient instant during the steady straight-line preliminary approach. Fig. A.2 shows the system of axes obtained.

The notation is standard (Ref. 4) and we have

$$\left.\begin{aligned} \gamma_e &= -\beta_m = -\phi + \hat{V} \tan\phi, \\ U_e &= V_e, \\ W_e &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

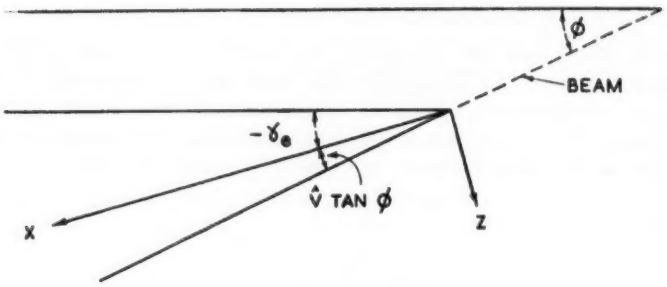

Fig. A.2.

3. Conditions of Ideal Flight

The ideal flight can be specified in terms of the velocity and attitude increments, (u, w, θ), from the equilibrium flight just defined. Then the elevator setting and thrust may be determined so that the aircraft may perform this required motion. The longitudinal equations of motion of an aeroplane are used here in assessing these ideal increments and, following Neumark[4], are written

$$\left.\begin{aligned} \frac{d}{d\tau}\left(\frac{u}{U_e}\right) - x_u\left(\frac{u}{U_e}\right) \quad - x_w\left(\frac{w}{U_e}\right) \quad + k_e(\theta) \quad - (x_T) = 0 \\ - z_u\left(\frac{u}{U_e}\right) + \frac{d}{d\tau}\left(\frac{w}{U_e}\right) - z_w\left(\frac{w}{U_e}\right) \quad - \left(\frac{d\theta}{d\tau}\right) + k_e \tan\gamma_e(\theta) \quad = 0 \\ \kappa\left(\frac{u}{U_e}\right) + \chi\frac{d}{d\tau}\left(\frac{w}{U_e}\right) + \omega\left(\frac{w}{U_e}\right) + \left(\frac{d^2\theta}{d\tau^2}\right) + \nu\left(\frac{d\theta}{d\tau}\right) \quad + \delta(\eta) \quad = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Of the five increments occurring in these three equations two may apparently be chosen arbitrarily. One of these, however, is fixed by the previous specification that the aircraft c.g. remains on the beam thereby relating θ_1 and w_1, and for ease of analysis the other is taken to be a prescribed variation of incidence.

3.1. *Variation of the incidence*

The beam oscillates with a frequency n. It is assumed, therefore, that in flying ideally down the beam the incidence is prescribed to oscillate about its steady, preliminary approach value, with the same frequency, so that the small incidence variations can be written

$$\alpha_1 = \alpha_e + k\sin n(\tau+\tau_1) + l\cos n(\tau+\tau_1) \quad . \quad . \quad . \quad (6)$$

Then, since $\dfrac{w}{U_e}$ = incidence increment,

the expression for w_1/U_e is determined immediately from equation (6) as

$$\frac{w_1}{U_e} = k \sin n(\tau+\tau_1) + l \cos n(\tau+\tau_1) \quad . \quad . \quad . \quad . \quad (7)$$

3.2. *Variation of the attitude*

The motion of the aircraft c.g. is such that it remains on the oscillating beam at all times. The constant, preliminary approach c.g. velocity vector has been taken to define the location of the x-axis in the body and its equilibrium direction. Assuming for the moment no variation of the incidence, the x-axis remains along the c.g. velocity vector, which is now variable, and a variation of attitude ensues, θ_1 say. A further contribution, θ_2 say, arises from the incidence changes so that the total attitude change is $\theta_1 = \theta_1 + \theta_2$.

Now θ is measured positively for an increase in climbing angle and thus

$$\theta_1 = -(\beta - \beta_m) = c \cos n(\tau+\tau_1).$$

θ_2 is simply equal to the incidence change so that the complete expression for θ_1 becomes

$$\theta_1 = k \sin n(\tau+\tau_1) + (l+c) \cos n(\tau+\tau_1). \quad . \quad . \quad . \quad . \quad (8)$$

3.3. *Variation of the velocity*

Having now found expressions for w_1 and θ_1, the second equation of motion (equations (5)) is used to determine u_1. Introducing the portmanteau notation of Neumark[4] this is

$$\frac{u_1}{U_e} = \frac{1}{z_u} \{ [k(N_1 - Q_1) + cn] \sin n(\tau+\tau_1) + [l(N_1 - Q_1) + ck_e \tan \gamma_e] \cos n(\tau+\tau_1) \}$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

3.4. *Variations of thrust and elevator settings*

The ideal thrust and elevator increments x_{T1} and η_1 are obtained directly by the substitution of u_1, w_1, θ_1 in the first and third of equations (5). They are

$$x_{T1} = \frac{1}{z_u} \{ [k(p_1 - R_1) - ln(N_1 - Q_1) + cn Q_1] \sin n(\tau+\tau_1) +$$
$$+ [kn(N_1 - Q_1) + l(p_1 - R_1) + c(n^2 - R_1)] \cos n(\tau+\tau_1) \} \quad . \quad (10)$$

$$\eta_1 = \frac{1}{\delta} \{ [k(n^2 - \omega) + ln(\nu + \chi) + cn\nu] \sin n(\tau+\tau_1) -$$
$$- [kn(\nu + \chi) - l(n^2 - \omega) - cn^2] \cos n(\tau+\tau_1) \} \quad . \quad . \quad (11)$$

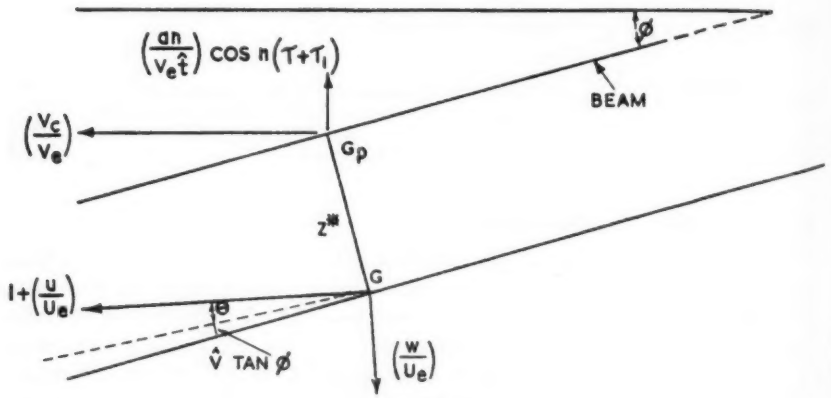

Fig. A.3.

4. Relation Between Track Error and Flight Increments

At any instant the position and dimensionless velocity of the aircraft c.g. relative to the beam are as shown in Fig. A.3, where G is the actual c.g. position and G_p is its projection upon the beam, so that G_pG is the track error z^* (now in dimensionless units)

Then $dz^*/d\tau$ is the dimensionless velocity of G relative to G_p in direction G_pG, which may be shown to be

$$\frac{dz^*}{d\tau} = \frac{w}{U_e} - \theta + c\cos n(\tau+\tau_1) \quad . \quad . \quad . \quad . \quad (12)$$

5. Equations of Controlled Motion Down the Beam

The controlled equations of motion are obtained by combining the equations of auto-control (equation (3) of the main paper and equation (12) of this Appendix) with the three equations of motion (5). These six equations may be simplified by the use of a certain transformation relating the actual and ideal motions and defining their difference as "perturbations." Thus, denoting these by * in the following scheme, we write

$$\left.\begin{aligned} u &= u_i + u^* U_e & \theta &= \theta_i + \theta^* \\ w &= w_i + w^* U_e & x_I &= x_{Ii} + x_I^* \\ & & \eta &= \eta_i + \eta^* \end{aligned}\right\} \quad . \quad . \quad . \quad (13)$$

It is clear that the ideal motion satisfies the equations of motion (5) and thus a simple substitution of equations (13) leads to the same set (5) with u^* replacing u/U_e, and so on. Similarly the control equations (equations (3) of the main paper) simplify into relations between the perturbations, and likewise equation (12) becomes

explicitly independent of the ideal motion. A slight rearrangement of these simplified equations leads to the following set giving the controlled motion down the beam.

$$\left.\begin{aligned}
&\frac{du^*}{d\tau}+e_u u^*+e_{dw/dt}\frac{dw^*}{d\tau}+e_w w^*+e_{d\theta/dt}\frac{d\theta^*}{d\tau}+e_\theta\theta^* \qquad +e_{dz/dt}\frac{dz^*}{d\tau}+e_z z^*+e_{\bar z}\int z^*d\tau-x_T^*=0\\
&\frac{du^*}{d\tau}-x_u u^* \qquad -x_w w^* \qquad +k_c\theta^* \qquad -x_T^*=0\\
&-z_u u^*+\frac{dw^*}{d\tau}-z_w w^*-\frac{d\theta^*}{d\tau}+k_c\tan\gamma_0.\theta^* \qquad =0\\
&\kappa u^*+\chi\frac{dw^*}{d\tau}+\omega w^*+\frac{d^2\theta^*}{d\tau^2}+\nu\frac{d\theta^*}{d\tau}+\delta\eta^* \qquad =0\\
&\frac{du^*}{d\tau}-f_u u^*-f_{dw/dt}\frac{dw^*}{d\tau}-f_w w^*-f_{d\theta/dt}\frac{d\theta^*}{d\tau}-f_\theta\theta^*+\eta^*+f_{dz/dt}\frac{dz^*}{d\tau}+f_z z^*+f_{\bar z}\int z^*d\tau \qquad =0\\
&-w^* \qquad +\theta^* \qquad +\frac{dz^*}{d\tau} \qquad =0
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

These "perturbation" equations now contain no explicit reference to the ideal motion and thus the design of a satisfactory auto-pilot may be entirely removed from the remainder of the problem and treated as a separate entity. Any resulting conclusions would be of general validity; for example, the results could be applied directly to the case of an airfield blind approach for there the beam is stationary and no ideal motion exists.

APPENDIX II
Numerical Data

Aircraft Design Details

Wing loading $=40$ lb./ft.^{2}, $\frac{\partial C_{L_B}}{\partial\alpha}=1/29$, $i_B=1/12$.

$S=250$ ft.^{2}, $\frac{\partial C_L}{\partial\alpha}=4.5$, $\lambda=6$, $\frac{\partial\varepsilon}{\partial\alpha}=0.4$.

$S'/S=1/6$, $\bar c=6$ ft., $\frac{\partial C_{LT}}{\partial\alpha'}=3$, $\frac{\partial C_{LT}}{\partial\eta}=1.5$.

Length of tail arm $l_T=18$ ft.,

$C_L=C_L$ (flaps up) $+\Delta C_L$ (flaps) $=1.0+1.2=2.2$.

$C_D=C_{Df}+C_{Di}=0.15+0.28=0.43$.

$\mu=\frac{W_1}{g\rho Sl_T}=29$, $\hat t=\frac{W_1}{g\rho SV_e}=4.24$ sec.

Carrier and Beam System

$V_c=25$ knots, $a=10$ ft., $\frac{2\pi}{N}=8$ sec., $\phi=15$ deg. (low thrust).

$\phi=7.5$ deg. (high thrust).

Equilibrium Flight Conditions (Preliminary Approach)

$V_e = 73$ knots., $\alpha_e = 12$ deg., $\gamma_e = -9.8$ deg., $T_e = 233$ lb. (low thrust).

$\gamma_e = -4.9$ deg., $T_e = 1090$ lb. (high thrust).

Aerodynamic Derivatives

$\kappa = 0$, $\omega = 2$, $\nu = 3$, $\chi = 1.2$, $\delta = 43.5$.

$x_u = -0.46$ (low thrust), $x_u = -0.55$ high thrust).

$x_w = 0.52$, $z_u = -2.2$, $z_w = -2.465$, $k_e = 1.1$.

Other Details

Height above mean sea level at auto-pilot engagement	= 1,000 ft.
Mean height of deck above mean sea level	= 60 ft.
Ideal time of flight down the beam	= 87.5 sec.

N_1, P_1, Q_1, R_1, S_1, T_1	portmanteau symbols (*see* Ref. 4)	
e_u, $e_{du/dt}$, etc.	thrust control gearings	*See* equations (1) and (2)
f_u, $f_{du/dt}$, etc.	elevator control gearings	
x, y, z	wind axes of reference, fixed in body	
x_u, x_w, z_u, z_w	aerodynamic force derivatives (*see* Ref. 4)	
κ, ω, ν, χ, δ	aerodynamic modified moment derivatives (*see* Ref. 4)	
u^*, w^*, θ^*, x_T^*, η^*	perturbation variables (*see* equation (13) of Appendix)	

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Supersonic Flow over Thin Symmetrical Wings with Given Surface Pressure Distribution

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Summary: The inverse problem of determining the supersonic flow past a thin symmetrical wing at zero incidence with given surface pressure distribution is solved for wings of arbitrary plan form. Expressions are obtained for the vertical derivative of the potential on the wing surface from which, using the linearised boundary condition of zero normal velocity at the surface, the profile of the wing can be designed. The integral equation method adopted by J. C. Evvard and extended by G. N. Ward is used. The analysis cannot be applied to pointed wings, whose leading edges are subsonic. The results in Part I are completely general and are applied to specific problems in Part II.

1. Introduction

The inverse problem of designing the profile of a thin wing set at zero incidence to an initially uniform stream and with given pressure distribution on the surface is the last of a set of four problems to be solved in the linearised theory of supersonic flow past thin wings. The two direct problems of determining the flow past a thin wing of given profile have been treated by several authors, in particular by Puckett[1] and Evvard[2] for symmetrical flow, and by Ward[3] in the anti-symmetrical case (*i.e.* flow at incidence over flat plates). The determination of the flow in the former of these two problems is easy in principle since the boundary conditions over the whole plane, in which the wing lies, are not mixed. The same is also true for the inverse problem in anti-symmetrical flow. The inverse problem for the symmetrical wing, now under consideration, is more difficult since the boundary conditions are mixed (*i.e.* the pressure distribution on the wing and the component of the disturbance velocity normal to the rest of the wing plane are given).

The gas is assumed perfect and the flow taken to be inviscid, irrotational, and isentropic; then, neglecting the squares and higher powers of the disturbance velocities due to the presence of the wing, linearised theory may be applied. The resulting linearised equation of motion is solved by using a result due to Hadamard[4], and the wing profile is designed by using the linearised boundary condition at the wing surface.

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[The Aeronautical Quarterly, Volume III, February 1952]

The method of solution is similar to that used by Ward, integral equations being established to determine the velocity normal to the plane of the wing in terms of the given pressure distribution on the wing surface. Off the wing the normal velocity is known and the pressure distribution may be determined by solving further integral equations. By using the transformation to characteristic co-ordinates due to Evvard these expressions are written as integral equations of the Abel type[5] and therefore admit ease in solving.

Part II of this paper is concerned with the application of the general theory to two specific problems of designing the profiles of wings of rectangular and arrowhead plan forms, on whose surface linear pressure distributions are taken. Singularities enter into the solution of both problems and, in the case of the rectangular wing, can be explained by taking the tip edges to be rounded. The result for the arrowhead wing clearly demonstrates that discontinuous pressure gradients give rise to singular solutions.

Notation

x, y, z	Cartesian co-ordinates with the x-axis parallel to the direction of the main stream
U	velocity of the main stream
ρ_0	density of the main stream
M	Mach number of the main stream
B	defined by $B^2 = M^2 - 1$
u	disturbance velocity due to the presence of the wing
ϕ	velocity potential defined by equation (2)
x', y'	co-ordinates of the points in the area of integration given by inequality (4)
$z = Z(x, y)$	equation of the wing surface
p	(Part I) disturbed pressure over the wing surface
p	(Part II) Heaviside operator
α, β	characteristic co-ordinates given by $\alpha = x - By$, $\beta = x + By$
α', β'	characteristic co-ordinates given by $\alpha' = x' - By'$, $\beta' = x' + By'$
$L(\alpha, \beta)$	defined as $L(\alpha, \beta) = -(2B/\pi)\phi(x, y, +0)$
S	projected area of the wing onto the plane $z = 0$
C	boundary of area S
R	remainder of plane $z = 0$
O	origin of co-ordinates
α_0, β_0	point of intersection P (*see* Fig. 1)
$A_1(\alpha), B_1(\alpha)$	functions of α defined by equation (15)

$A_2(\beta), B_2(\beta)$	functions of β defined by equation (16)
$f(\alpha, \beta)$	the value of $L(\alpha, \beta)$ on S
g_1, g_2, h_1, h_2	the values of $L(\alpha, \beta)$ on special regions of R (*see* equations (18) and (19))
$F_1(\alpha), F_2(\alpha)$	functions introduced in equation (22)
ζ	parameter introduced in solving equation (23)
ξ, ξ', η'	variables defined by equation (41)
$x=a$	equation of leading edge of the rectangular wing
d	wing chord
$2l$	breadth of wing
C_1, D_1	constants in equation (37)
C_2, D_2	constants in equation (38)
C_3, D_3	constants in equation (49)
τ	angle of sweepback (*see* Fig. 5)
f_R, f_L	values of $f(\alpha, \beta)$ on the right side and the left side of the arrowhead wing respectively
$\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$	regions defined in Fig. 5

PART I
GENERAL THEORY

2. Fundamental Equations and Boundary Conditions

A wing, whose surface makes everywhere a small angle with the x- and y-axes so that it lies approximately in the plane $z=0$ in suitable Cartesian co-ordinates, is placed in a uniform supersonic stream parallel to the x-axis with velocity U, density ρ_0 and Mach number M. Neglecting viscosity and heat conduction the flow is practically isentropic (since the change of entropy at weak shocks is proportional to the cube of their strength) so the linearised equation of isentropic flow

$$B^2 \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

holds as a first approximation, where $B^2 = M^2 - 1$ and the disturbance velocity to the uniform stream is given by

$$\mathbf{u} = \nabla \phi \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equation (1) is a second order hyperbolic equation and the solution can be given in terms of data given on the plane $z=0$ in a case when $u=0$ for x less than

some constant. Assuming that the potential on $z=+0$, $\phi(x,y,0)$, is continuous and bounded, the solution of equation (1) for $z>0$, say, is

$$\phi(x,y,z)=-\frac{1}{\pi}\frac{\partial}{\partial z}\left[\iint\frac{\phi(x',y',+0)\,dx'\,dy'}{\sqrt{\{(x-x')^2-B^2(y-y')^2-B^2z^2\}}}\right], \quad . \quad (3)$$

where the integration is taken over the area of the plane $z=0$ for which

$$x'\leqslant x-B\sqrt{\{(y-y')^2+z^2\}} \quad . \quad . \quad . \quad . \quad (4)$$

Expression (3) is obtained by differentiating with respect to z the equation for $\int_0^z \phi\,dz$ in terms of the values of ϕ on $z=+0$. This latter equation is obtained by substitution of $\int_0^z \phi\,dz$ for ϕ in Hadamard's well-known result[4] expressing ϕ in terms of the values of $\partial\phi/\partial z$ on $z=+0$.

If the wing is symmetric, the disturbance potential and its x- and y-derivatives are even functions of z, and $\partial\phi/\partial z$ is an odd function of z. Let the equation of the wing surface be $z=Z(x,y)$, then the boundary condition of zero normal velocity at the upper surface, to a linear approximation, is

$$(\partial\phi/\partial z)_{z=+0}=\partial Z/\partial x. \quad . \quad . \quad . \quad . \quad (5)$$

On the rest of the plane $z=0$, $\partial\phi/\partial z$, as a continuous odd function of z, must be zero.

When the disturbed pressure distribution, p, is specified over the wing surface, as it will be in this case, integration with respect to x of the linearised Bernoulli equation

$$(\partial\phi/\partial z)_{z=+0}=\partial Z/\partial x. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

determines the value of $\phi(x,y,+0)$ over the wing surface.

Summarising, the disturbance potential $\phi(x,y,z)$ satisfies equation (1) and in the region $z>0$, say, has the following mixed boundary conditions:

(*i*) ϕ is continuous and bounded everywhere in $z\geqslant+0$;
(*ii*) $\phi=0$ ahead of some plane $x=$constant;
(*iii*) $\phi(x,y,+0)$ takes given values on the wing;
(*iv*) $\partial\phi/\partial z=0$ on the plane $z=0$, except over the wing. } . . (7)

Then, if $\partial\phi/\partial z$ is found over the wing, using condition (5) the wing can be designed to produce the required pressure distribution.

$\partial\phi/\partial z$ can best be calculated by differentiating equation (3) with respect to z:—

$$\frac{\partial\phi}{\partial z}=\frac{\partial^2}{\partial z^2}\left[-\frac{1}{\pi}\iint\frac{\phi(x',y',+0)\,dx'\,dy'}{\sqrt{\{(x-x')^2-B^2(y-y')^2-B^2z^2\}}}\right] \quad . \quad . \quad (8)$$

Since the quantity contained in the square brackets is the integral of ϕ with respect to z and therefore satisfies equation (1), equation (8) may be written

$$\frac{\partial\phi}{\partial z}=-\frac{1}{\pi}\left(B^2\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\left[\iint\frac{\phi(x',y',+0)\,dx'\,dy'}{\sqrt{\{(x-x')^2-B^2(y-y')^2-B^2z^2\}}}\right] \quad . \quad (9)$$

Now taking the special case of equation (9) when $z=+0$, the value of $\partial\phi/\partial z$ on $z=+0$ is given by

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=+0}=-\frac{1}{\pi}\left(B^2\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\left[\iint\frac{\phi(x',y',+0)\,dx'\,dy'}{\sqrt{\{(x-x')^2-B^2(y-y')^2\}}}\right] \quad (10)$$

the integration being taken over the area of the plane $z=0$ for which

$$x'\leqslant x-B|y-y'|. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The form of expression (10) is appropriate for the application of the transformation into characteristic co-ordinates (α,β) given by

$$\alpha=x-By, \quad \beta=x+By. \quad . \quad . \quad . \quad . \quad (12)$$

Putting $(2B/\pi)\phi(x',y',+0)=-L(\alpha',\beta')$ and making the transformation (12), equation (10) becomes

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=+0}=\frac{\partial^2}{\partial\alpha\,\partial\beta}\left[\iint\frac{L(\alpha',\beta')\,d\alpha'\,d\beta'}{\sqrt{\{(\alpha-\alpha')(\beta-\beta')\}}}\right], \quad . \quad . \quad (13)$$

the limits of integration being

$$\alpha'\leqslant\alpha, \quad \beta'\leqslant\beta. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

In the manipulation of equation (13) it is convenient to divide the wing surface as shown in Fig. 1.

Let the projection of the wing on to the plane $z=0$ be S, with boundary C, and denote the remainder of the plane $z=0$ by R. In Fig. 1, L_1P_1, L_2P_2, T_1U, T_2U are characteristic lines, α or β=constant, for equation (1) in the (x,y) plane and tangents to the curve C at points L_1, L_2, T_1, T_2 respectively. L_1P, L_2P, T_1U_1, T_2U_2 are also characteristic lines. The origin of co-ordinates is taken at O, the point of intersection of L_1P_1 and L_2P_2, the co-ordinates of L_1, L_2 being $(\alpha_0, 0)$, $(0, \beta_0)$ respectively. The curvature of C is assumed to have the same sign everywhere

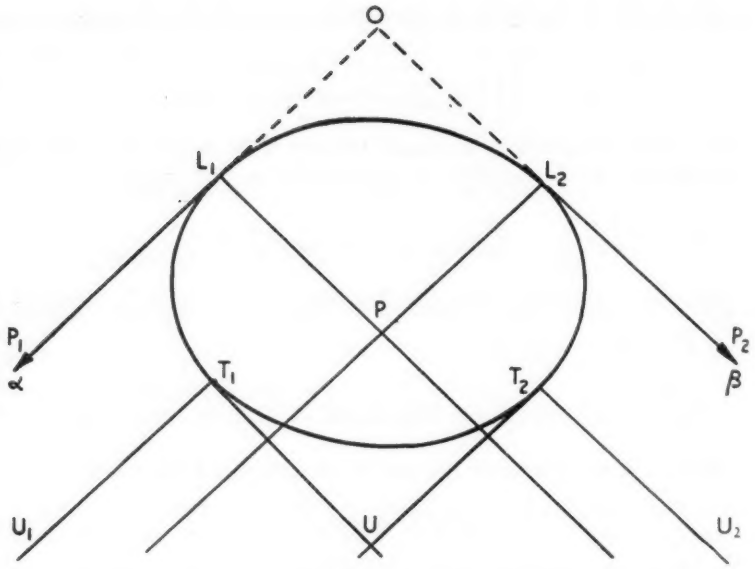

Fig. 1.

(*i.e.* C is convex), so that it may be defined in terms of single-valued functions:—

$$\text{On } T_1 L_1 L_2,\ \beta = A_1(\alpha); \qquad \text{on } L_2 T_2 T_1,\ \beta = A_2(\alpha), \quad . \quad . \quad (15)$$

or alternatively:—

$$\text{On } L_1 L_2 T_2,\ \alpha = B_1(\beta); \qquad \text{on } T_2 T_1 L_1,\ \alpha = B_2(\beta). \quad . \quad . \quad (16)$$

Integral equations are now set up to determine $\partial\phi/\partial z$ on S, this being done in intermediate stages.

3. The General Wing

A general case, as indicated in Fig. 2, is considered immediately.
The problem is fully solved without real difficulty when the wing has aspect ratio sufficiently large for the subsonic edges $L_1 T_1$ and $L_2 T_2$ to be independent in the sense that the characteristic lines $L_1 P$, $L_2 P$ intersect C behind them, *i.e.* in the supersonic trailing edge $T_1 T_2$. A special case of this is when the wing-tip flows are independent, *i.e.* when the characteristic lines $L_1 P$, $L_2 P$ intersect at a point P lying outside S. If the aspect ratio is too small the flow is theoretically determinable, but difficulty in solving the integral equations is encountered.

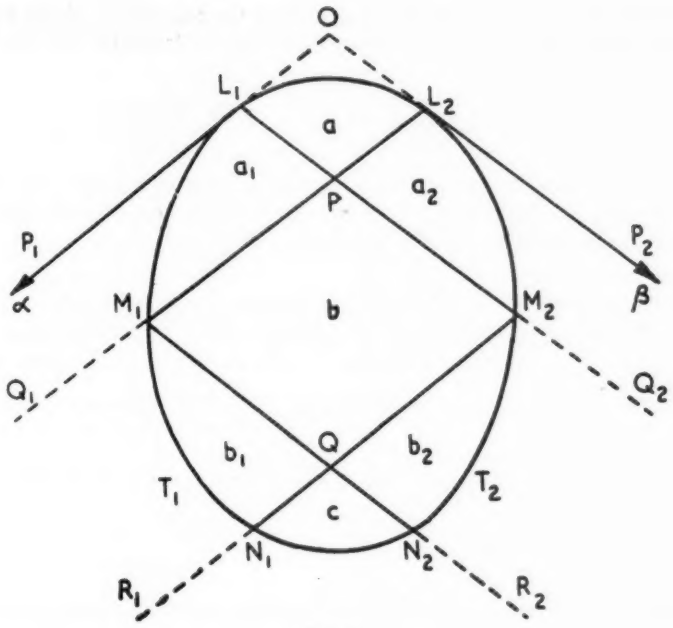

Fig. 2.

The range of influence of the disturbance due to the wing on the plane $z=0$ is the region downstream from $P_1 L_1 L_2 P_2$. Elsewhere on the plane $z=0$ the potential vanishes identically. Hence integration in equation (13) need only be taken over that part of the region given by inequalities (14) which lies downstream from $P_1 L_1 L_2 P_2$.

For the wing plan form shown in Fig. 2, let

$$L(\alpha,\beta)=f\ (\alpha,\beta) \text{ on } S; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\left.\begin{aligned} &=g_1(\alpha,\beta) \text{ in region } P_1 L_1 M_1 Q_1; \\ &=g_2(\alpha,\beta) \text{ in region } P_2 L_2 M_2 Q_2; \end{aligned}\right\} \quad . \quad . \quad . \quad (18)$$

$$\left.\begin{aligned} &=h_1(\alpha,\beta) \text{ in region } Q_1 M_1 N_1 R_1; \\ &=h_2(\alpha,\beta) \text{ in region } Q_2 M_2 N_2 R_2, \end{aligned}\right\} \quad . \quad . \quad . \quad (19)$$

where functions $g_1(\alpha,\beta)$, $h_1(\alpha,\beta)$, and so on are, as yet, unknown.

(*i*) For a point (α, β) on S in region a (*see* Fig. 2) the z-derivative of the potential is obtained from equation (13) by inserting limits derived from (15) and (16) and is

$$\left(\frac{\partial \phi}{\partial z}\right)_{z=+0} = \frac{\partial^2}{\partial \alpha \, \partial \beta}\Big[\int_{B_1(\beta)}^{\alpha} \frac{d\alpha'}{\sqrt{(\alpha-\alpha')}} \int_{A_1(\alpha')}^{\beta} \frac{f(\alpha', \beta')\, d\beta'}{\sqrt{(\beta-\beta')}}\Big]. \quad . \quad . \quad . \quad (20)$$

(*ii*) For a point in region $P_2 L_2 M_2 Q_2$ (*i.e.* $0<\alpha<\alpha_0$; $\beta>A_2(\alpha)$) we have from equation (13) and condition (7) (*iv*)

$$0 = \frac{\partial^2}{\partial \alpha \, \partial \beta}\Big[\int_0^{\alpha} \frac{d\alpha'}{\sqrt{(\alpha-\alpha')}} \Big\{\int_{A_1(\alpha')}^{A_2(\alpha')} \frac{f(\alpha', \beta')\, d\beta'}{\sqrt{(\beta-\beta')}} + \int_{A_2(\alpha')}^{\beta} \frac{g_2(\alpha', \beta')\, d\beta'}{\sqrt{(\beta-\beta')}}\Big\}\Big] \quad . \quad (21)$$

Hence

$$\int_0^{\alpha} \frac{d\alpha'}{\sqrt{(\alpha-\alpha')}} \Big\{\int_{A_1(\alpha')}^{A_2(\alpha')} \frac{f(\alpha', \beta')\, d\beta'}{\sqrt{(\beta-\beta')}} + \int_{A_2(\alpha')}^{\beta} \frac{g_2(\alpha', \beta')\, d\beta'}{\sqrt{(\beta-\beta')}}\Big\} = F_1(\alpha) + F_2(\beta), \quad . \quad (22)$$

for some functions $F_1(\alpha)$, $F_2(\beta)$.

Equation (22) is an integral equation of the Abel type for the quantity contained in the curly brackets. By condition (7) (*i*) f and g_2 are bounded; hence the expression in the curly brackets is bounded and $F_1(0)+F_2(\beta)=0$. The solution under these conditions is

$$\int_{A_1(\alpha)}^{A_2(\alpha)} \frac{f(\alpha, \beta')\, d\beta'}{\sqrt{(\beta-\beta')}} + \int_{A_2(\alpha)}^{\beta} \frac{g_2(\alpha, \beta')\, d\beta'}{\sqrt{(\beta-\beta')}} = \frac{1}{\pi} \int_0^{\alpha} \frac{F_1'(\alpha')\, d\alpha'}{\sqrt{(\alpha-\alpha')}}. \quad . \quad . \quad (23)$$

Letting $\beta \longrightarrow A_2(\alpha)$, the right hand side is determined as

$$\frac{1}{\pi} \int_0^{\alpha} \frac{F_1'(\alpha')\, d\alpha'}{\sqrt{(\alpha-\alpha')}} = \int_{A_1(\alpha)}^{A_2(\alpha)} \frac{f(\alpha, \beta')\, d\beta'}{\sqrt{\{A_2(\alpha)-\beta'\}}}. \quad . \quad . \quad . \quad . \quad (24)$$

Equation (23), together with result (24), forms an Abel integral equation for $g_2(\alpha, \beta)$ with solution

$$g_2(\alpha, \beta) = \frac{1}{\pi} \frac{\partial}{\partial \beta}\Big[\int_{A_2(\alpha)}^{\beta} \frac{d\zeta}{\sqrt{(\beta-\zeta)}} \Big\{\int_{A_1(\alpha)}^{A_2(\alpha)} \frac{f(\alpha, \beta')\, d\beta'}{\sqrt{\{A_2(\alpha)-\beta'\}}} - \int_{A_1(\alpha)}^{A_2(\alpha)} \frac{f(\alpha, \beta')\, d\beta'}{\sqrt{(\zeta-\beta')}}\Big\}\Big]. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

From equation (25) the final expression for $g_2(\alpha, \beta)$ is determined as

$$g_2(\alpha,\beta)=\frac{\sqrt{\{\beta-A_2(\alpha)\}}}{\pi}\int_{A_1(\alpha)}^{A_2(\alpha)}\frac{f(\alpha,\beta')d\beta'}{(\beta-\beta')\sqrt{\{A_2(\alpha)-\beta'\}}}. \qquad (26)$$

It can easily be verified that the potential is continuous at $\beta=A_2(\alpha)$ in the sense that

$$\lim_{\beta\to A_1(\alpha)} g_2(\alpha,\beta)=f(\alpha,A_2(\alpha)). \qquad (27)$$

Thus the value of the potential in region $P_2L_2M_2Q_2$ is given by expression (26) and is continuous at the edges of the wing. Similarly the potential at a point in the region $P_1L_1M_1Q_1$ (*see* Fig. 2) can be determined.

(*iii*) Using expressions (13), $\partial\phi/\partial z$ at a point (α, β) in region a_2 of S (*i.e.* $0<\alpha<\alpha_0$; $\beta_0<\beta<A_2(\alpha)$) is given by

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=+0}=\frac{\partial^2}{\partial\alpha\,\partial\beta}\Bigg[\int_{B_1(\beta)}^{\alpha}\frac{d\alpha'}{\sqrt{(\alpha-\alpha')}}\int_{A_1(\alpha')}^{\beta}\frac{f(\alpha',\beta')d\beta'}{\sqrt{(\beta-\beta')}}+$$
$$+\int_0^{B_1(\beta)}\frac{d\alpha'}{\sqrt{(\alpha-\alpha')}}\left\{\int_{A_1(\alpha')}^{A_2(\alpha')}\frac{f(\alpha',\beta')d\beta'}{\sqrt{(\beta-\beta')}}+\int_{A_1(\alpha')}^{\beta}\frac{g_2(\alpha',\beta')d\beta'}{\sqrt{(\beta-\beta')}}\right\}\Bigg]. \qquad (28)$$

Applying equations (23) and (24), the latter term of this equation can be replaced by a term involving the known function, $f(\alpha, \beta)$, only; thus equation (28) becomes

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=+0}=\frac{\partial^2}{\partial\alpha\,\partial\beta}\Bigg[\int_{B_1(\beta)}^{\alpha}\frac{d\alpha'}{\sqrt{(\alpha-\alpha')}}\int_{A_1(\alpha')}^{\beta}\frac{f(\alpha',\beta')d\beta'}{\sqrt{(\beta-\beta')}}+\int_0^{B_1(\beta)}\frac{d\alpha'}{\sqrt{(\alpha-\alpha')}}\int_{A_1(\alpha')}^{A_2(\alpha')}\frac{f(\alpha',\beta')d\beta'}{\sqrt{\{A_2(\alpha')-\beta'\}}}\Bigg], \qquad (29)$$

or at points where $\partial B_1(\beta)/\partial\beta$ is continuous,

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=+0}=\frac{\partial}{\partial\alpha}\int_{B_1(\beta)}^{\alpha}\frac{d\alpha'}{\sqrt{(\alpha-\alpha')}}\frac{\partial}{\partial\beta}\int_{A_1(\alpha')}^{\beta}\frac{f(\alpha',\beta')d\beta'}{\sqrt{(\beta-\beta')}}, \qquad (30)$$

since $A_2\{B_1(\beta)\}=\beta$.

A similar expression can be obtained for a point in region a_1 (Fig. 2), namely

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=+0}=\frac{\partial}{\partial\beta}\int_{A_1(\alpha)}^{\beta}\frac{d\beta'}{\sqrt{(\beta-\beta')}}\frac{\partial}{\partial\alpha}\int_{B_1(\beta')}^{\alpha}\frac{f(\alpha',\beta')d\alpha'}{\sqrt{(\alpha-\alpha')}}. \qquad (31)$$

When the flows over the wing-tips are independent and the pressure distribution on the wing surface is given, then the z-derivative of the potential can be calculated everywhere on S using expressions (20), (30), and (31), and the profile of the wing determined. If this is not the case then further calculations are necessary, as follows:—

(*iv*) For a point (α, β) on S in region b expression (13) becomes:—

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=+0} = \frac{\partial^2}{\partial\alpha\,\partial\beta}\Bigg[\int_{B_1(\beta)}^{\alpha}\frac{d\alpha'}{\sqrt{(\alpha-\alpha')}}\int_{A_1(\alpha)}^{\beta}\frac{f(\alpha',\beta')\,d\beta'}{\sqrt{(\beta-\beta')}} - \int_{B_1\{A_1(\alpha)\}}^{B_1(\beta)}\frac{d\alpha'}{\sqrt{(\alpha-\alpha')}}\int_{A_1(\alpha')}^{A_1(\alpha)}\frac{f(\alpha',\beta')\,d\beta'}{\sqrt{(\beta-\beta')}} +$$

$$+\int_{0}^{B_1(\beta)}\frac{d\alpha'}{\sqrt{(\alpha-\alpha')}}\left\{\int_{A_1(\alpha)}^{A_2(\alpha')}\frac{f(\alpha',\beta')\,d\beta'}{\sqrt{(\beta-\beta')}} + \int_{A_2(\alpha')}^{\beta}\frac{g_2(\alpha',\beta')\,d\beta'}{\sqrt{(\beta-\beta')}}\right\} +$$

$$+\int_{0}^{A_1(\alpha)}\frac{d\beta'}{\sqrt{(\beta-\beta')}}\left\{\int_{B_1(\beta)}^{B_2(\beta')}\frac{f(\alpha',\beta')\,d\alpha'}{\sqrt{(\alpha-\alpha')}} + \int_{B_2(\beta')}^{\alpha}\frac{g_1(\alpha',\beta')\,d\alpha'}{\sqrt{(\alpha-\alpha')}}\right\}\Bigg]. \qquad (32)$$

Of the integrals in equation (32) the first is taken over the lower shaded rectangle in Fig. 3, the second over the upper shaded rectangle; the others cover the remainder of the dependence domain, and each also contains the upper shaded rectangle. The third integral occurs in equation (28) and was simplified to become the second term in equation (29); similarly the fourth integral can be simplified. Using these results, expression (32) becomes:—

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=+0} = \frac{\partial^2}{\partial\alpha\,\partial\beta}\Bigg[\int_{B_1(\beta)}^{\alpha}\frac{d\alpha'}{\sqrt{(\alpha-\alpha')}}\int_{A_1(\alpha)}^{\beta}\frac{f(\alpha',\beta')\,d\beta'}{\sqrt{(\beta-\beta')}} - \int_{B_1\{A_1(\alpha)\}}^{B_1(\beta)}\frac{d\alpha'}{\sqrt{(\alpha-\alpha')}}\int_{A_1(\alpha')}^{A_1(\alpha)}\frac{f(\alpha',\beta')\,d\beta'}{\sqrt{(\beta-\beta')}}\Bigg] -$$

$$-2\left(\frac{\partial^2}{\partial\alpha\,\partial\beta}\sqrt{\{\alpha - B_1(\beta)\}}\right)\int_{A_1\{B_1(\beta)\}}^{\beta}\frac{f(B_1(\beta),\beta')\,d\beta'}{\sqrt{(\beta-\beta')}} -$$

$$-2\left(\frac{\partial^2}{\partial\alpha\,\partial\beta}\sqrt{\{\beta - A_1(\alpha)\}}\right)\int_{B_1\{A_1(\alpha)\}}^{\alpha}\frac{f(\alpha',A_1(\alpha))\,d\alpha'}{\sqrt{(\alpha-\alpha')}}. \qquad (33)$$

Thus $(\partial\phi/\partial z)_{z=+0}$ is given by expression (13) taken over the lower shaded area, and (with the sign changed) over the upper shaded area, plus two line integrals along the characteristic lines which intersect at the point K (which has co-ordinates $B_1(\beta), A_1(\alpha)$) in region a and bound the areas.

The first two terms of expression (33) are analogous to the result obtained by Ward[3] for the potential at a point on a wing of zero thickness and given shape.

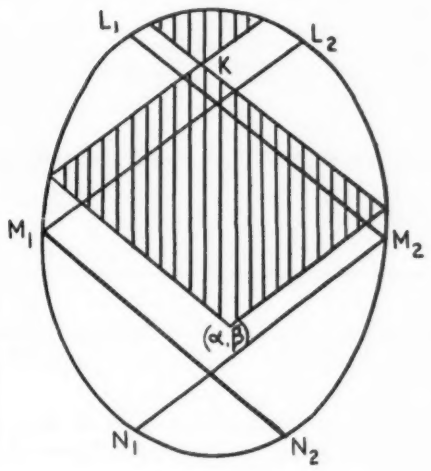

Fig. 3.

It is evident from Fig. 2 that, if the flows round the subsonic edges are independent, regions a, a_1, a_2, b cover the whole wing; thus expression (33) and the preceeding ones determine the value of $\partial\phi/\partial z$ everywhere on the wing surface. No case where this is not so will be worked out. An indication is given in (*v*), which follows, how the calculation could be progressively continued in such cases by methods analogous to those that have gone before.

(*v*) For a point (α, β) in region $Q_2 M_2 N_2 R_2$ (Fig. 2) we have from equation (13) and condition (7) (*iv*)

$$0=\frac{\partial^2}{\partial\alpha\,\partial\beta}\Big[\int_0^{\alpha_2}\frac{d\alpha'}{\sqrt{(\alpha-\alpha')}}\left\{\int_{A_1(\alpha')}^{A_2(\alpha')}\frac{f(\alpha',\beta')\,d\beta'}{\sqrt{(\beta-\beta')}}+\int_{A_2(\alpha')}^{\beta}\frac{g_2(\alpha',\beta')\,d\beta'}{\sqrt{(\beta-\beta')}}\right\}+$$

$$+\int_{\alpha_2}^{\alpha}\frac{d\alpha'}{\sqrt{(\alpha-\alpha')}}\left\{\int_0^{A_1(\alpha')}\frac{g_1(\alpha',\beta')\,d\beta'}{\sqrt{(\beta-\beta')}}+\int_{A_1(\alpha')}^{A_2(\alpha')}\frac{f(\alpha',\beta')\,d\beta'}{\sqrt{(\beta-\beta')}}+\int_{A_2(\alpha')}^{\beta}\frac{h_2(\alpha',\beta')\,d\beta'}{\sqrt{(\beta-\beta')}}\right\}\Big]. \quad (34)$$

The first term of this equation is zero, for the expression in the curly brackets, by equations (23) and (24), is independent of β. An Abel integral equation remains from which, using the method of (*ii*) of this Section, the potential $h_2(\alpha, \beta)$ in the

region under consideration can be found. The resultant expression is

$$h_2(\alpha,\beta)=\frac{\sqrt{\{\beta-A_2(\alpha)\}}}{\pi}\left\{\int_{A_1(\alpha)}^{A_2(\alpha)}\frac{f(\alpha,\beta')\,d\beta'}{(\beta-\beta')\sqrt{\{A_2(\alpha)-\beta'\}}}+\int_{0}^{A_1(\alpha)}\frac{g_1(\alpha',\beta')\,d\beta'}{(\beta-\beta')\sqrt{\{A_2(\alpha)-\beta'\}}}\right\} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Similarly the potential in region $Q_1 M_1 N_1 R_1$ can be determined and is

$$h_1(\alpha,\beta)=\frac{\sqrt{\{\alpha-B_2(\beta)\}}}{\pi}\left\{\int_{B_1(\beta)}^{B_2(\beta)}\frac{f(\alpha',\beta)\,d\alpha'}{(\alpha-\alpha')\sqrt{\{B_2(\beta)-\alpha'\}}}+\int_{0}^{B_1(\beta)}\frac{g_2(\alpha',\beta')\,d\alpha'}{(\alpha-\alpha')\sqrt{\{B_2(\beta)-\alpha'\}}}\right\} \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Since $g_1(\alpha,\beta)$, $g_2(\alpha,\beta)$ are known functions from equation (26) and the corresponding equation for the left wing tip, $h_1(\alpha,\beta)$, $h_2(\alpha,\beta)$ are determined from equations (35) and (36).

By application of equation (13) the value of $\partial\phi/\partial z$ for a point in any one of the regions b_1, b_2 and c (Fig. 2) can be expressed in terms of known functions.

If N_1 or N_2 lies outside $T_1 T_2$, the regions of the wing lettered in Fig. 2 do not cover it completely and the procedure indicated in (*ii*), (*iii*) and (*iv*) of this Section has to be repeated yet again, systematically finding the value of ϕ off the wing, then $\partial\phi/\partial z$ on the wing surface. It is seen that for wings of very small aspect ratio the method will be extremely laborious.

In the limiting case when the wing is pointed and the leading edges lie inside the Mach cone from the apex (*i.e.* are "subsonic") the points L_1 and L_2 coincide and the method cannot be applied at all.

PART II

SPECIFIC PROBLEMS

4. The Rectangular Wing

In this section a wing of rectangular plan form, with leading edge normal to the direction of the undisturbed flow, is considered; the isobars are taken parallel to the leading edge and along any vertical section of the wing the pressure varies linearly.

Let the wing have chord of length d and breadth $2l$ such that the ratio $(d/l)<B$ (*i.e.* independent wing-tip flows). Taking the isobars to be lines $x=$constant and using equation (6) together with the given linear pressure law, the potential at the wing surface is found to be of the form

$$\phi=C_1 x^2+D_1 x+f(y), \quad . \quad . \quad . \quad . \quad . \quad (37)$$

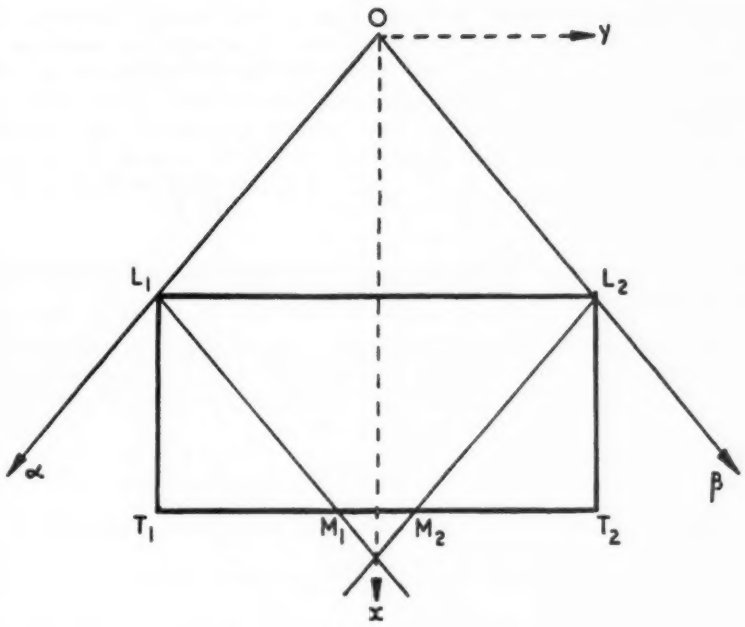

Fig. 4.

where C_1, D_1 are known constants and $f(y)$ is a function determined from the condition that the potential vanishes at the leading edge $x=a$.

The final expression for the potential on the wing surface becomes

$$\phi = C_2(x-a)^2 + D_2(x-a), \quad . \quad . \quad . \quad . \quad . \quad (38)$$

or in characteristic co-ordinates,

$$\phi = (C_2/4)(\alpha+\beta-2a)^2 + (D_2/2)(\alpha+\beta-2a), \quad . \quad . \quad . \quad (39)$$

where C_2 and D_2 are known constants.

Using expression (20), $\partial\phi/\partial z$ at a point (α, β) in region $L_1\, M_1\, M_2\, L_2$ (Fig. 4), is

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=+0} = -\frac{B}{\pi}\frac{\partial^2}{\partial\alpha\,\partial\beta}\left[\int_{2a-\beta}^{\alpha}\frac{d\alpha'}{\sqrt{(\alpha-\alpha')}}\int_{2a-\alpha'}^{\beta}\frac{[(C_2/2)(\alpha'+\beta'-2a)+D_2(\alpha'+\beta'-2a)]\,d\beta'}{\sqrt{(\beta-\beta')}}\right]. \quad . \quad . \quad . \quad . \quad . \quad (40)$$

By substituting new variables given by

$$\eta' = \alpha' + \beta' - 2a, \quad \xi' = \alpha' + \beta - 2a, \quad \xi = \alpha + \beta - 2a, \qquad (41)$$

expression (40) reduces to:—

$$\left(\frac{\partial \phi}{\partial z}\right)_{z=+0} = -\frac{B}{\pi}\frac{\partial^2}{\partial \xi^2}\int_0^{\xi} \frac{d\xi'}{\sqrt{(\xi - \xi')}} \int_0^{\xi'} \frac{[(C_2/2)\eta'^2 + D_2\eta']\, d\eta'}{\sqrt{(\xi' - \eta')}}, \qquad (42)$$

which is an integral of the Faltung type, so that Heaviside operational methods may be applied:—

$$\left(\frac{\partial \phi}{\partial z}\right)_{z=+0} = -\frac{B}{\pi} p^2 \left[\frac{1}{p}\frac{(-\frac{1}{2})!}{p^{-\frac{1}{2}}}\frac{1}{p}\frac{(-\frac{1}{2})!}{p^{-\frac{1}{2}}}\left\{\frac{C_2}{2}\frac{2!}{p^2} + \frac{D_2}{p}\right\}\right], \qquad (43)$$

$$= -B\{2C_2(\alpha + \beta - 2a) + D_2\}, \qquad (44)$$

$$= -B\{2C_2(x - a) + D_2\}. \qquad (45)$$

Thus the equation of the wing surface, using condition (5), is given by

$$z = Z(x, y) = -B\{C_2 x^2 - 2C_2 ax + D_2 x\} + g(y), \qquad (46)$$

where $g(y)$ is a function to be determined from the condition that the wing closes up at the leading edge. The wing must also close up at the trailing edge thus imposing a condition between the constants C_2 and D_2. Using these results expression (46) becomes

$$z = BC_2(x - a)(a + d - x), \qquad (47)$$

which checks with ordinary two-dimensional theory.

For a point in $L_2M_2T_2$ the value of $\partial\phi/\partial z$ obtained from expression (30) is

$$\left(\frac{\partial \phi}{\partial z}\right)_{z=+0} = -\frac{2B}{\pi}\left[\{2C_2(x - a) + D_2\}\tan^{-1}\sqrt{\left\{\frac{B(l - y)}{(x - a) - B(l - y)}\right\}} + \right.$$

$$\left. + \left(\frac{2}{3}C_2\{2(x - a) + B(l - y)\} + D_2\right)\sqrt{\left\{\frac{x - a}{B(l - y)} - 1\right\}}\right]. \qquad (48)$$

As the wing edge L_2T_2 is approached, it is seen that $\partial\phi/\partial z$ behaves singularly like the inverse square root of the distance from the edge. This implies that the wing surface has a vertical tangent there (*i.e.* the tip edges are rounded). The linearised theory, as given in Part I of this paper, cannot be expected to hold at the rounded edges; but the implication that the tip edges are rounded is perhaps more than

could have been expected from a linearised theory, for it is in good agreement with the design requirements of subsonic edges. Thus, from equation (5) the wing surface can be designed everywhere outside a small neighbourhood of the subsonic edges, L_1T_1, L_2T_2, where there is a vertical tangent. Alternatively the pressure distribution may be suitably modified near the edges so that the integral of expression (30) yields a solution for $\partial\phi/\partial z$ which is regular at the wing tip edges (*i.e.* the tip edges are then tapered).

5. The Arrowhead Wing

A wing of arrowhead plan form (bounded by L_1OL_2, T_1TT_2 in Fig. 5) is considered when the edges are everywhere supersonic. The pressure on the wing surface is so distributed that the isobars are parallel to the leading edge and the pressure varies linearly with the normal distance from the leading edge.

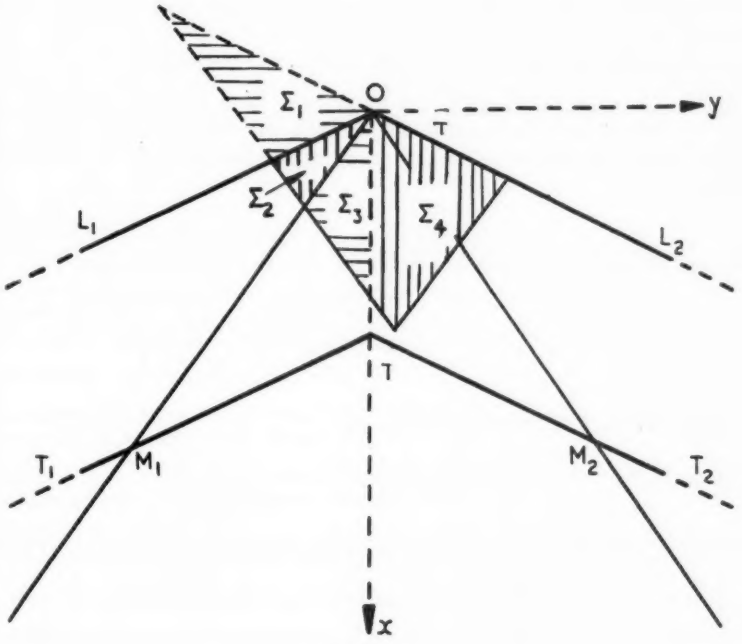

Fig. 5.

Let the angle of sweepback of the wing be τ such that $\tan\tau < B$. The isobars are lines $x\cos\tau \pm y\sin\tau = \text{constant}$, the positive sign being taken for the left wing

tip. In a similar manner to the method indicated in Section 4 the final expression for the potential on the wing surface is

$$\phi = C_3 (x \cos\tau \pm y \sin\tau)^2 + D_3 (x \cos\tau \pm y \sin\tau), \quad . \quad . \quad (49)$$

or in characteristic co-ordinates,

$$\phi = \{ C_3/(4B^2) \} \{ \alpha (B\cos\tau \mp \sin\tau) + \beta (B\cos\tau \pm \sin\tau) \}^2 + \\ + \{ D_3/(2B) \} \{ \alpha (B\cos\tau \mp \sin\tau) + \beta (B\cos\tau \pm \sin\tau) \}, \quad (50)$$

where C_3 and D_3 are known constants and when multiplied by $\cos\tau$ correspond to C_2 and D_2 used in Section 4.

$\partial\phi/\partial z$ at a point (x, y) in region $L_2 OM_2 T_2$ can be evaluated from expression (20) using the method of Section 4, or the problem treated as a rectangular wing whose leading edge is at an angle $\frac{1}{2}\pi - \tau$ to the stream and result (45) used.

Thus $$(\partial\phi/\partial z)_{z=+0} = -\sqrt{\{ B^2\cos^2\tau - \sin^2\tau \}} [2C_3 (x\cos\tau - y\sin\tau) + D_3], \quad . \quad (51)$$

from which, using condition (5), the wing surface may be designed.

$\partial\phi/\partial z$ at a point (α, β) inside the Mach cone from the apex is rather more difficult to evaluate and the method adopted will only be indicated. For a point in region OTM_2 the limits of integration of expression (20) can be divided (*see* Fig. 5):—

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=+0} = \frac{\partial^2}{\partial\alpha\,\partial\beta}\Bigg[\iint\limits_{\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4} \frac{f_R(\alpha',\beta')\,d\alpha'\,d\beta'}{\sqrt{\{(\alpha-\alpha')(\beta-\beta')\}}} + \iint\limits_{\Sigma_3} \frac{[f_L(\alpha',\beta') - f_R(\alpha',\beta')]\,d\alpha'\,d\beta'}{\sqrt{\{(\alpha-\alpha')(\beta-\beta')\}}} + \\ + \iint\limits_{\Sigma_4} \frac{f_L(\alpha',\beta')\,d\alpha'\,d\beta'}{\sqrt{\{(\alpha-\alpha')(\beta-\beta')\}}} - \iint\limits_{\Sigma_1+\Sigma_2} \frac{f_R(\alpha',\beta')\,d\alpha'\,d\beta'}{\sqrt{\{(\alpha-\alpha')(\beta-\beta')\}}} \Bigg], \\ . \quad . \quad . \quad . \quad . \quad (52)$$

from which the final expression is found as

$$(\partial\phi/\partial z)_{z=+0} = -\sqrt{\{ B^2\cos^2\tau - \sin^2\tau \}} [2C_3 (x\cos\tau - y\sin\tau) + D_3] + \\ + [(2C_3 \sin 2\tau)/\pi] \sqrt{\{x^2 - B^2 y^2\}} + \\ + \{ [4(2C_3 x\cos\tau + D_3)\sin\tau]/\pi \} \log [\sqrt{(2By)}/\{\sqrt{(x-By)} + \sqrt{(x+By)}\}] + \\ + (2\sqrt{\{ B^2\cos^2\tau - \sin^2\tau \}}/\pi) \Big[\{ 2C_3 (x\cos\tau - y\sin\tau) + \\ + D_3 \} \sin^{-1}\sqrt{\{ (B\cos\tau + \sin\tau)(x-By)/(x\cos\tau - y\sin\tau) \}} - \\ - \{ 2C_3 (x\cos\tau + y\sin\tau) + \\ + D_3 \} \sin^{-1}\sqrt{\{ (B\cos\tau - \sin\tau)(x-By)/(x\cos\tau + y\sin\tau) \}} \Big]. \\ . \quad . \quad . \quad . \quad . \quad (53)$$

As the x-axis is approached $\partial\phi/\partial z$ becomes infinite, suggesting that pressure distributions with discontinuous gradients yield singular solutions. Since the singularity behaves like $\log y$ as $y \longrightarrow 0$, it is possible from equations (5), (51) and (53) to design the wing surface everywhere, except for a small neighbourhood of the line *OT* (*see* Fig. 5) along which the pressure gradient is discontinuous. A surface of reasonable shape, continuous with the theoretically determined surface, may now be chosen to complete the wing design across the neighbourhood of the singularity. This would give approximately the required pressure distribution outside the neighbourhood and a modified distribution inside. Alternatively by making the pressure gradient continuous across the x-axis in the theoretical considerations, the singularity would probably not arise.

6. Further Problems

Other pressure laws could be studied so that singularities which have arisen in the previous two cases do not occur. Another problem, which is of interest, comes from taking the isobars, obeying a similar pressure law as in the present case, to radiate from the vertices at the ends of the trailing edge of a triangular wing. This problem has been studied by the author and, although an answer could be obtained (by numerical methods, if necessary), it does not seem to admit a result in terms of elementary functions.

Acknowledgment

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Longitudinal Waves in Beams

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Summary: The elementary engineering theory of longitudinal waves in beams allows for no dispersion, whereas Love's corrected version of it predicts that very short waves are not propagated. A new correction is added which overcomes these defects. To do this, a simple theorem is proved concerning certain types of closed region. A method of improving Love's formula and the present one is illustrated.

1. Introduction

When problems involving the flexural motion of beams are investigated, use may be made of the well-known simple theory (*see*, for example, Ref. 11, Section 54). If greater accuracy is required (especially as regards short waves), two more exact versions are available. These are presented by Rayleigh[8] and Timoshenko[9] as corrected forms of the most elementary theory in which two of its known shortcomings are rectified. As Timoshenko[10] shows, his equation is remarkably accurate.* These three may be termed "engineering theories" and they owe their importance to the fact that the *exact* results of dynamical elasticity are far too complicated for any but the simplest problems. All of the engineering formulae are obtainable from elementary (strength of materials) reasoning and all tend to exactitude as the waves concerned become longer.

The state of affairs is not quite as happy for longitudinal vibrations in beams. Here, again, exact theory is too intractable for most practical applications and it is necessary to turn to engineering theories. The simple theory is well known (*see*, for example, Ref. 11, Section 51) and it approaches agreement with the exact one for long waves. But its prediction of a fixed propagation speed ($=\sqrt{(E/\rho)}$) for all wave lengths is contrary to the exact results and to what is actually observable, as found by Davies[4]. This simple theory has an obvious defect in that no regard is paid to lateral inertia forces, although lateral displacements must necessarily occur.

A correction to overcome this is offered by Love[5] and his results show good agreement with those of the exact equations over a wider range of wave length than before. However, Love's equation indicates that short waves are transmitted at vanishingly small speeds and this is neither compatible with the true results nor with physical intuition. A less important point perhaps, but one worthy of consideration, is that this correction is obtained from Hamilton's Principle and is

*See also the work of Prescott[7] and Christopherson[2].

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thus unable to take its place with all the other equations in the average book on strength of materials.

Notation

A	area of cross section
A'	area defined in Section 3 (*see* Fig. 2)
a	radius of cylinder
c	phase propagation speed
c_0	phase propagation speed of long waves $[=\sqrt{(E/\rho)}]$
c_2	speed of propagation of distortion waves in the infinite medium $[=\sqrt{(\mu/\rho)}]$
c_s	speed of propagation of surface waves
E	Young's modulus
k_p	polar radius of gyration of cross section
$\left.\begin{matrix} r,\theta \\ R,\Theta \end{matrix}\right\}$	cylindrical co-ordinates (*see* Fig. 2)
P,Q	defined by equations (19)
R,S	defined by equations (21)
t	time
u,v,w	displacements
x,y,z	rectangular co-ordinates
y_0,z_0	maximum values of y and z (Fig 1)
$\sigma_{xx},\sigma_{yy},\sigma_{zz}$	normal stresses
$\tau_{xy},\tau_{yz},\tau_{zx}$	shear stresses
Λ	wave length
μ	modulus of rigidity: one of Lamé's constants $[=E/\{2(1+\nu)\}]$
ν	Poisson's ratio
ξ	wave number $(=2\pi/\Lambda)$
ρ	density
$\varphi(\theta_1,\theta)$	function defined in Section 3.

2. New Theory

The simplest theory of longitudinal motions of beams rests upon the assumption that, while vibrating, planes perpendicular to the axis remain plane. No restraint is allowed on the lateral motion of particles but the inertia forces

associated with it are neglected. Love's correction is concerned with these forces and a correction for them is also devised here.

Consider a prismatic beam whose cross-sectional area is A and whose centroidal axis is identified with the x-axis, and contemplate a thin slice of the beam of

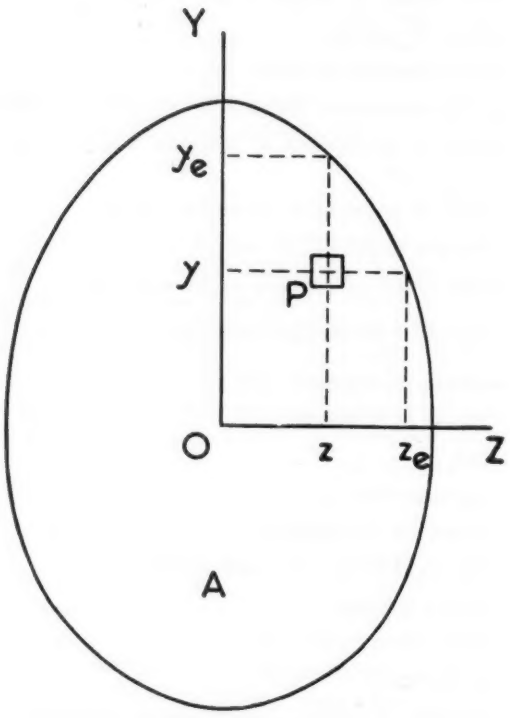

Fig. 1.

thickness dx (*see* Fig. 1). Applying the law of motion to a small element $dxdydz$ at P in the y and z directions, two of the equations of motion are obtained, *i.e.* in the usual notation,

$$\left.\begin{aligned} \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= \rho \frac{\partial^2 v}{\partial t^2}, \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= \rho \frac{\partial^2 w}{\partial t^2}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

These can be simplified if it is assumed that plane sections remain plane and that, as a first estimate, the displacements in the y and z directions are

$$v = -\nu y \frac{\partial u}{\partial x}; \quad w = -\nu z \frac{\partial u}{\partial x}. \qquad (2)$$

These are Love's assumptions; but he omits consideration of shear stresses. With μ as the modulus of rigidity,

$$\left.\begin{aligned} \tau_{xy} &= \mu \frac{\partial v}{\partial x} = -\nu\mu y \frac{\partial^2 u}{\partial x^2}, \\ \tau_{xz} &= \mu \frac{\partial w}{\partial x} = -\nu\mu z \frac{\partial^2 u}{\partial x^2}, \\ \tau_{yz} &= 0. \end{aligned}\right\} \qquad (3)$$

Substituting into (1) and integrating the first equation with respect to y and the second with respect to z,

$$\left.\begin{aligned} \sigma_{yy} &= \frac{\nu\rho}{2}(y_e^2 - y^2)\frac{\partial^3 u}{\partial x \partial t^2} - \frac{\nu\mu}{2}(y_e^2 - y^2)\frac{\partial^3 u}{\partial x^3}, \\ \sigma_{zz} &= \frac{\nu\rho}{2}(z_e^2 - z^2)\frac{\partial^3 u}{\partial x \partial t^2} - \frac{\nu\mu}{2}(z_e^2 - z^2)\frac{\partial^3 u}{\partial x^3}. \end{aligned}\right\} \qquad (4)$$

The quantities y_e, z_e represent the maximum values of y and z corresponding to P as shown in Fig. 1; in general, they will be functions of the co-ordinates (y, z) of P.

Now, at P

$$E\frac{\partial u}{\partial x} = \sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz}) \qquad (5)$$

so that

$$\sigma_{xx} = E\frac{\partial u}{\partial x} + \frac{\nu^2\rho}{2}[(y_e^2 + z_e^2) - (y^2 + z^2)]\frac{\partial^3 u}{\partial x \partial t^2} - \frac{\nu^2\mu}{2}[(y_e^2 + z_e^2) - (y^2 + z^2)]\frac{\partial^3 u}{\partial x^3} \qquad (6)$$

Applying the law of motion to the thin slice in the x direction then,

$$d\left[\int_A \sigma_{xx}\, dA\right] = A\, dx\, \rho \frac{\partial^2 u}{\partial t^2}$$

or

$$\int_A \frac{\partial \sigma_{xx}}{\partial x} dA = A\rho \frac{\partial^2 u}{\partial t^2} \qquad (7)$$

On substituting from (6) into (7) and carrying out the indicated differentiation and integration,

$$A\rho\frac{\partial^2 u}{\partial t^2}=EA\frac{\partial^2 u}{\partial x^2}+\nu^2\rho\frac{\partial^4 u}{\partial x^2\partial t^2}\int_A \tfrac{1}{2}[(y_e^2+z_e^2)-(y^2+z^2)]\,dA-$$

$$-\nu^2\mu\frac{\partial^4 u}{\partial x^4}\int_A \tfrac{1}{2}[(y_e^2+z_e^2)-(y^2+z^2)]\,dA \quad . \quad (8)$$

This equation may be taken as that of longitudinal motion of the beam, the functions y_e and z_e being introduced for any particular cross section. Obviously (8) would be a most difficult equation to work with for any but the simplest sections. However, it can be greatly simplified if use is made of the following mathematical result (which will be demonstrated in Section 3):—

$$\int_A (y_e^2+z_e^2)\,dA=3\int_A (y^2+z^2)\,dA \quad . \quad . \quad . \quad (9)$$

If this be accepted temporarily, the equation of motion may be written

$$\frac{\partial^2 u}{\partial t^2}=c_0^2\frac{\partial^2 u}{\partial x^2}+\nu^2k_p^2\frac{\partial^4 u}{\partial x^2\partial t^2}-\nu^2k_p^2c_2^2\frac{\partial^4 u}{\partial x^4} \quad . \quad . \quad . \quad (10)$$

where c_0 is the "bar velocity" $[=\sqrt{(E/\rho)}]$, k_p is the polar radius of gyration of the cross section A and c_2 is the speed of distortion waves in an infinite elastic medium $[=\sqrt{(\mu/\rho)}]$.

If, in the foregoing arguments, all stresses other than σ_{xx} had been neglected, the elementary equation of long waves would have emerged. In point of fact, it is equation (10) with the two fourth derivatives omitted.

Next, Love's equation can be obtained by contemplating only σ_{xx}, σ_{yy} and σ_{zz}. It is identical with (10) if the term involving $\partial^4 u/\partial x^4$ is excluded, as may be seen by tracing through the analysis.

3. Mathematical Theorem

Consider the area A' which is bounded by the axes OY, OZ and a curve C which is given by $y=F(z)$, where $F(z)$ is a single-valued function of z which decreases monotonically as z increases (so that $F'(z)\leqslant 0$ for $z\geqslant 0$). Suppose that C is represented by $r=f(\theta)$ in polar co-ordinates where $f(\theta)$ is single-valued in θ. Let P be some point in A' with co-ordinates (r,θ) and let the perpendiculars from P to the axes intersect C as shown in Fig. 2, at P_1 and P_2 (with co-ordinates (r_1,θ_1) and (r_2,θ_2) respectively). Let the perpendicular to OY through P_1 intersect that to OZ through P_2 in P' whose co-ordinates are (R,Θ) (*see* Fig. 2).

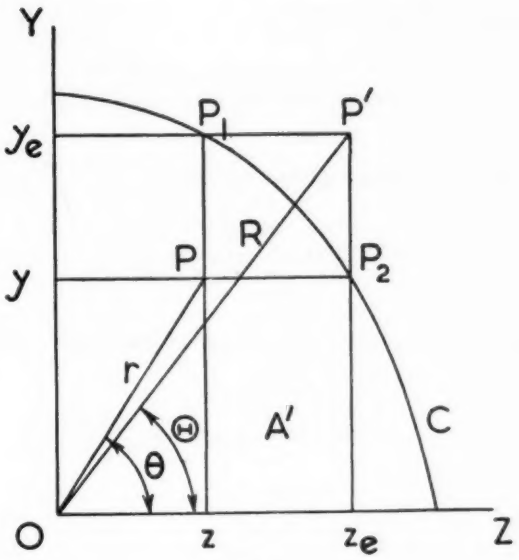

Fig. 2.

It is to be proved that, if dA' is the element of area surrounding P, then

$$\int_{A'} R^2 dA' = 3 \int_{A'} r^2 dA'.$$

Since $dA' = r dr d\theta$ and the equation of C is $r = f(\theta)$, it follows that

$$\int_{A'} r^2 dA' = \int_0^{\pi/2} d\theta \int_0^{f(\theta)} r^3 dr = \tfrac{1}{4} \int_0^{\pi/2} [f(\theta)]^4 d\theta .$$

Now $r_1 = f(\theta_1)$ so that

$$f(\theta_1) \cos\theta_1 = r \cos\theta ,$$

and $r_2 = f(\theta_2)$ so that

$$f(\theta_2) \sin\theta_2 = r \sin\theta .$$

Therefore $$R^2 = r_1^2 \sin^2\theta_1 + r_2^2 \cos^2\theta_2 = r_1^2 + r_2^2 - r^2$$

and $$\int_{A'} R^2 dA' = \int_{A'} (r_1^2 + r_2^2) dA' - \int_{A'} r^2 dA'.$$

That is, it remains to be shown that

$$\int_{A'} r_1^2 dA' + \int_{A'} r_2^2 dA' = 4 \int_{A'} r^2 dA' = \int_0^{\pi/2} [f(\theta)]^4 d\theta .$$

Consider the first of these integrals. Evidently,

$$\int_{A'} r_1^2 dA' = \int_0^{\pi/2} d\theta \int_0^{f(\theta)} r [f(\theta_1)]^2 dr .$$

Now $r = f(\theta_1) \cos\theta_1 / \cos\theta$ so that, as r varies from 0 to $f(\theta)$, θ_1 varies steadily from $\pi/2$ to θ. Thus,

$$dr = [f'(\theta_1) \cos\theta_1 - f(\theta_1) \sin\theta_1] \frac{d\theta_1}{\cos\theta},$$

so that

$$\int_{A'} r_1^2 dA' = \int_0^{\pi/2} \sec^2\theta \, d\theta \int_{\pi/2}^{\theta} [f(\theta_1)]^3 [f'(\theta_1) \cos\theta_1 - f(\theta_1) \sin\theta_1] \cos\theta_1 \, d\theta_1$$

$$= \int_0^{\pi/2} [f(\theta_1)]^3 [f(\theta_1) \sin\theta_1 - f'(\theta_1) \cos\theta_1] \cos\theta_1 \, d\theta_1 \int_0^{\theta_1} \sec^2\theta \, d\theta .$$

That the last change of limits and of the order of integration is valid will be seen by reference to Fig. 3, the vertical axis being that of the function $\varphi(\theta_1, \theta)$ where

$$\varphi(\theta_1, \theta) = [f(\theta_1)]^3 [f(\theta_1) \sin\theta_1 - f'(\theta_1) \cos\theta_1] \cos\theta_1 \sec^2\theta .$$

It will be noted that both double integrals represent the volume over the shaded area. Therefore,

$$\int_{A'} r_1^2 dA' = \int_0^{\pi/2} [f(\theta)]^3 [f(\theta) \sin\theta - f'(\theta) \cos\theta] \sin\theta \, d\theta ,$$

the suffix 1 being dropped.

The integral

$$\int_{A'} r_2^2 dA' = \int_0^{\pi/2} d\theta \int_0^{f(\theta)} r [f(\theta_2)]^2 dr$$

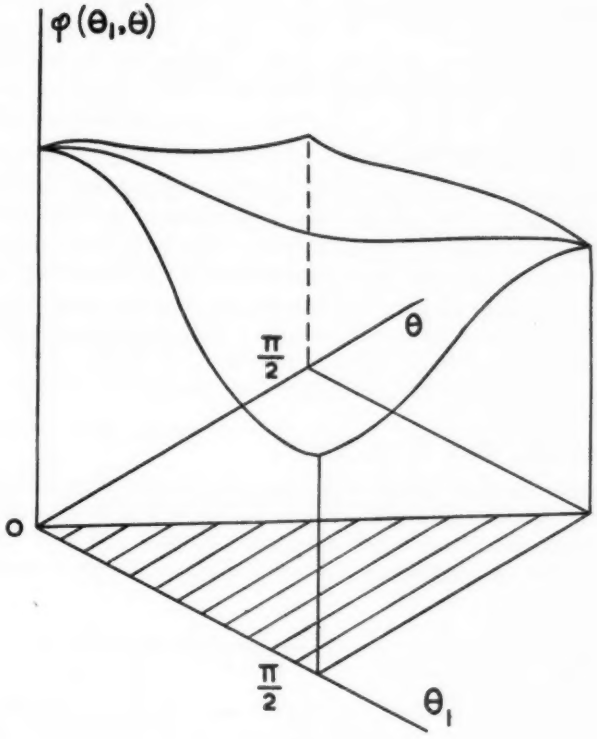

Fig. 3.

can be treated in a similar way. Using the relation $r = f(\theta_2)\sin\theta_2/\sin\theta$, this becomes

$$\int_{A'} r_2^2 dA' = \int_0^{\pi/2} \operatorname{cosec}^2\theta\, d\theta \int_0^{\theta} [f(\theta_2)]^3 [f'(\theta_2)\sin\theta_2 + f(\theta_2)\cos\theta_2]\sin\theta_2\, d\theta_2$$

$$= \int_0^{\pi/2} [f(\theta_2)]^3 [f'(\theta_2)\sin\theta_2 + f(\theta_2)\cos\theta_2]\sin\theta_2\, d\theta_2 \int_{\theta_2}^{\pi/2} \operatorname{cosec}^2\theta\, d\theta$$

by the same sort of reasoning as before. Thus

$$\int_{A'} r_2^2 dA' = \int_0^{\pi/2} [f(\theta)]^3 [f'(\theta)\sin\theta + f(\theta)\cos\theta]\cos\theta\, d\theta\,,$$

the suffix 2 now being dispensed with. It is now clear that

$$\int_{A'} r_1^2 dA' + \int_{A'} r_2^2 dA' = \int_0^{\pi/2} [f(\theta)]^4 d\theta$$

so that the theorem is proved.

At this point a word might be said about the beam cross sections which are covered by this theory; for restrictions have been placed upon the curve C. The value of the corrections for lateral inertia which have been discussed seems questionable for sections other than those allowed in the proof. For example, no allowance can be made for the fact that the inner flange surfaces of an I-beam remain stress-free during longitudinal motion. However, it may happen that the *empirical* use of the equations is justified even for such sections as this. Of course, no difficulty arises with square-, rectangular- or circular-cross sections.

4. Comparison with Exact Theory

The most convenient theory with which to compare these results is that of Pochhammer[6] referring to a cylindrical shaft; but, even for such a simple section as this, the dispersion equation (governing wave length Λ and wave speed c) offers an unpleasant task of computation. However, Bancroft[1] gives a satisfactory table

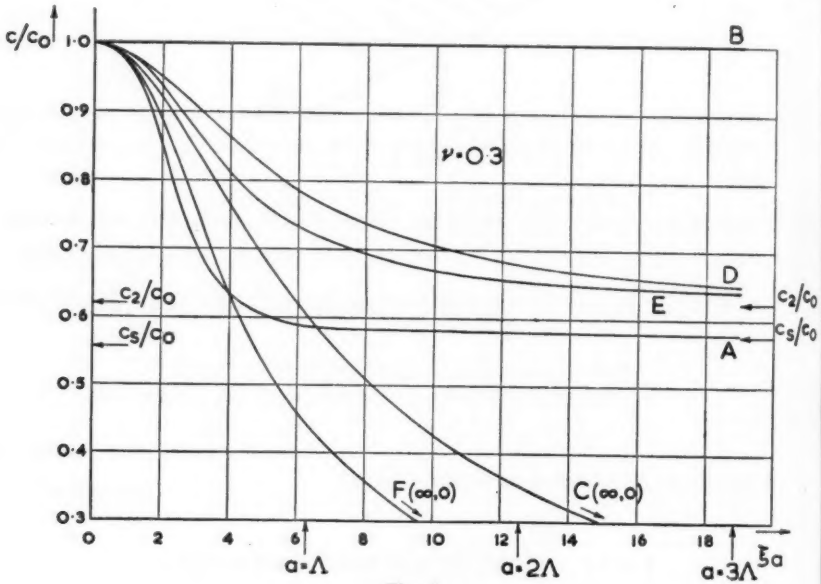

Fig. 4.

of numerical values of c/c_0 and (effectively) $a\xi$, a being the radius and $2\pi/\xi$ the wave length; this he does for several values of ν. Taking $\nu=0.3$, his results are shown in Fig. 4 as curve A and this will be used as the standard of comparison for the engineering formulae. The curve has a horizontal asymptote corresponding to c_s, the speed of surface waves.

This exact theory gives rise to an infinite number of higher modes of vibration whereby nodal cylinders appear in the longitudinal motion. The analytical meaning of these has been examined by Cooper[3] although they do not seem to have much practical importance (*see* Ref. 4). They have no counterpart in the engineering theories so that only the fundamental mode A need be considered.

The dispersion equation appropriate to the simplified theories may be found by trying a solution of the form

$$u=e^{i\xi(x-ct)} \qquad . \quad . \quad . \quad . \quad . \quad (11)$$

in the various equations of motion. This leads to the relations

$$\frac{c}{c_0}=1 \qquad \text{(elementary theory)}, \qquad . \quad . \quad (12)$$

$$\frac{c^2}{c_0^2}=\frac{1}{1+\nu^2k_p^2\xi^2} \qquad \text{(Love's theory)}, \qquad . \quad . \quad (13)$$

$$\frac{c^2}{c_0^2}=\frac{1+\nu^2k_p^2\xi^2c_2^2/c_0^2}{1+\nu^2k_p^2\xi^2} \qquad \text{(present theory)}. \qquad . \quad . \quad (14)$$

These are plotted as curves B, C and D in Fig. 4, respectively. For this purpose k_p^2 is replaced by $a^2/2$ and the value $\nu=0.3$ is used so that $c_2^2/c_0^2=[2(1+\nu)]^{-1}=0.385$.

Perhaps the most noticeable property of the curves is that D not only exhibits dispersion, but also has a horizontal asymptote of finite value. That of the exact theory corresponds to $c/c_0=c_s/c_0=0.575$ while the one for equation (14) is $c/c_0=c_2/c_0=0.621$ when $\nu=0.3$, an error of 8 per cent. Simple theory is not likely to lead to any better agreement than this because surface waves find no place in the theory of strength of materials.

One aspect of the new curve (D) may warrant further refinement; for while it is a great improvement upon B for all wave lengths, it is inferior to C for long waves. If this should be desirable (on account of the physical nature of the problem to be handled by this theory), the method of Section 5 may be used.

It is pointed out by Pochhammer[6] that, if all higher powers of $a\xi$ than the second are rejected from the exact theory, the wave speed for long waves is given by

$$\frac{c^2}{c_0^2}=\left(1-\frac{\nu^2a^2\xi^2}{2}\right) \qquad . \quad . \quad . \quad . \quad . \quad (13a)$$

This coincides with Love's theory, as may easily be found from equation (13).* If this degree of approximation is assumed in (14), however, we find

$$\frac{c^2}{c_0^2} = \left[1 - \frac{\nu^2 a^2 \xi^2}{2}\left(1 - \frac{c_2^2}{c_0^2}\right)\right] \quad . \quad . \quad . \quad . \quad (14a)$$

so that a new term appears (which is always less than half the one preceding it).

5. Improvement of Corrected Theories

In searching for a means of improving the engineering formulae, it seems logical to try to improve equations (2). They are obtained by integrating the relations

$$\left.\begin{aligned} E\frac{\partial v}{\partial y} &= \sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz}), \\ E\frac{\partial w}{\partial z} &= \sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy}), \end{aligned}\right\} \quad . \quad . \quad . \quad (15)$$

under the assumption that σ_{yy} and σ_{zz} are small compared with σ_{xx} and can therefore be neglected. From this, the three stresses were calculated (equations (4) and (6)).

Better agreement can be obtained by using the values of σ_{xx}, σ_{yy} and σ_{zz} (found in (4) and (6)) in equations (15) so that improved expressions for the displacements v and w emerge on integration. If this is done,

$$\left.\begin{aligned} v &= -\nu y\frac{\partial u}{\partial x} + \left(y_e^2 y - \frac{y^3}{3}\right)P + (z_e^2 - z^2)yQ, \\ w &= -\nu z\frac{\partial u}{\partial x} + \left(z_e^2 z - \frac{z^3}{3}\right)P + (y_e^2 - y^2)zQ, \end{aligned}\right\} \quad . \quad . \quad (16)$$

$$\left.\begin{aligned} \tau_{xy} &= -\nu\mu y\frac{\partial^2 u}{\partial x^2} + \mu\left(y_e^2 y - \frac{y^3}{3}\right)\frac{\partial P}{\partial x} + \mu(z_e^2 - z^2)y\frac{\partial Q}{\partial x}, \\ \tau_{xz} &= -\nu\mu z\frac{\partial^2 u}{\partial x^2} + \mu\left(z_e^2 z - \frac{z^3}{3}\right)\frac{\partial P}{\partial x} + \mu(y_e^2 - y^2)z\frac{\partial Q}{\partial x}, \end{aligned}\right\} \quad . \quad . \quad (17)$$

$$\left.\begin{aligned} \sigma_{yy} &= \frac{\nu}{2}(y_e^2 - y^2)\left[\rho\frac{\partial^3 u}{\partial x \partial t^2} - \mu\frac{\partial^3 u}{\partial x^3}\right] - \frac{1}{12}(y^4 - 6y_e^2 y^2 + 5y_e^4)\left[\rho\frac{\partial^2 P}{\partial t^2} - \mu\frac{\partial^2 P}{\partial x^2}\right] - \\ &\quad - \tfrac{1}{2}(z_e^2 - z^2)(y_e^2 - y^2)\left[\rho\frac{\partial^2 Q}{\partial t^2} - \mu\frac{\partial^2 Q}{\partial x^2}\right], \\ \sigma_{zz} &= \frac{\nu}{2}(z_e^2 - z^2)\left[\rho\frac{\partial^3 u}{\partial x \partial t^2} - \mu\frac{\partial^3 u}{\partial x^3}\right] - \frac{1}{12}(z^4 - 6z_e^2 z^2 + 5z_e^4)\left[\rho\frac{\partial^2 P}{\partial t^2} - \mu\frac{\partial^2 P}{\partial x^2}\right] - \\ &\quad - \tfrac{1}{2}(y_e^2 - y^2)(z_e^2 - z^2)\left[\rho\frac{\partial^2 Q}{\partial t^2} - \mu\frac{\partial^2 Q}{\partial x^2}\right], \end{aligned}\right\} \quad (18)$$

*It must not be concluded that Love's theory is a second approximation of the exact one in any other sense, however.

where

$$\left.\begin{aligned} P &= \frac{\nu(1-\nu^2)}{2c_0^2}\left[\frac{\partial^3 u}{\partial x \partial t^2} - c_2^2 \frac{\partial^3 u}{\partial x^3}\right], \\ Q &= -\frac{\nu^2(1+\nu)}{2c_0^2}\left[\frac{\partial^3 u}{\partial x \partial t^2} - c_2^2 \frac{\partial^3 u}{\partial x^3}\right]. \end{aligned}\right\} \quad . \quad . \quad . \quad (19)$$

Proceeding as for equations (5), (6) and (7), the improved equation of motion is

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial x^2} + \nu^2 k_p^2 \frac{\partial^4 u}{\partial x^2 \partial t^2} - \nu^2 k_p^2 c_2^2 \frac{\partial^4 u}{\partial x^4} - \\ - \frac{\nu}{12}\left(\frac{\partial^3 P}{\partial x \partial t^2} - c_2^2 \frac{\partial^3 P}{\partial x^3}\right) R - \nu\left(\frac{\partial^3 Q}{\partial x \partial t^2} - c_2^2 \frac{\partial^3 Q}{\partial x^3}\right) S, \quad . \quad (20)$$

where

$$\left.\begin{aligned} R &= \frac{1}{A}\int_A (y^4 - 6y_c^2 y^2 + 5y_c^4 + z^4 - 6z_c^2 z^2 + 5z_c^4)\, dA, \\ S &= \frac{1}{A}\int_A (y_c^2 - y^2)(z_c^2 - z^2)\, dA. \end{aligned}\right\} \quad . \quad . \quad (21)$$

Since no general result comparable to (9) can be found now, the particular case of a cylindrical shaft will be taken. For this cross section,

$$R = 4a^4, \quad S = a^4/3. \quad . \quad . \quad . \quad . \quad (22)$$

Thus (20) becomes, here,

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial x^2} + \frac{\nu^2 a^2}{2} \frac{\partial^4 u}{\partial x^2 \partial t^2} - \frac{\nu^2 a^2 c_2^2}{2} \frac{\partial^4 u}{\partial x^4} + \\ + \frac{\nu^2 a^4}{6c_0^2}(2\nu^2 + \nu - 1)\left[\frac{\partial^6 u}{\partial x^2 \partial t^4} - 2c_2^2 \frac{\partial^6 u}{\partial x^4 \partial t^2} + c_2^4 \frac{\partial^6 u}{\partial x^6}\right] \quad . \quad (23)$$

[The newly added correction vanishes if the material is incompressible (*i.e.* if $\nu = 1/2$)]. The corresponding dispersion equation is

$$\frac{c^4}{c_0^4} a^4 \xi^4 \frac{\nu^2}{6}(2\nu^2 + \nu - 1) - \frac{c^2}{c_0^2}\left[1 + \frac{\nu^2 a^2 \xi^2}{2} + \frac{\nu^2 c_2^2}{3c_0^2}(2\nu^2 + \nu - 1) a^4 \xi^4\right] + \\ + \left[1 + \frac{\nu^2 c_2^2}{2c_0^2} a^2 \xi^2 + \frac{\nu^2 c_2^4}{6c_0^4}(2\nu^2 + \nu - 1) a^4 \xi^4\right] = 0, \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and it is shown in Fig. 4 as curve *E*.

There is no reason why this sort of improvement should not be made on Love's equation. Remembering that all shear stresses must now be disregarded, the

equations may easily be shown to be exactly as before except that all terms containing μ and c_2^2 are omitted. Thus, the general equation (comparable with (20)) is

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial x^2} + \nu^2 k_p^2 \frac{\partial^4 u}{\partial x^2 \partial t^2} - \frac{\nu^2(1-\nu^2)}{24 c_0^2} \frac{\partial^6 u}{\partial x^2 \partial t^4} R + \\ + \frac{\nu^3(1+\nu)}{2c_0^2} \frac{\partial^6 u}{\partial x^2 \partial t^4} S \quad . \quad . \quad . \quad . \quad . \quad (25)$$

For a cylindrical shaft (*cf.* equation (23)),

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial x^2} + \frac{\nu^2 a^2}{2} \frac{\partial^4 u}{\partial x^2 \partial t^2} + \frac{\nu^2 a^4}{6 c_0^2} (2\nu^2 + \nu - 1) \frac{\partial^6 u}{\partial x^2 \partial t^4}, \quad . \quad (26)$$

and the dispersion equation is

$$\frac{c^4}{c_0^4} a^4 \xi^4 \frac{\nu^2}{6} (2\nu^2 + \nu - 1) - \frac{c^2}{c_0^2} \left[1 + \frac{\nu^2 a^2 \xi^2}{2} \right] + 1 = 0 \quad . \quad . \quad (27)$$

This last result is plotted as curve *F* in Fig. 4.

The curves show better agreement with the exact one. This is strikingly so with Love's equation over quite a wide range of (long) wave lengths. This fact might be used if the distortion of a pressure pulse (as it is propagated) is being studied by consideration of the first few terms of its Fourier representation.

6. Conclusions

The engineering theories of longitudinal waves in beams have been examined in terms of the simple notions of strength of materials; a derivation of Love's correction has been given which is thought to be simpler than the original one (at least for the non-mathematically minded) and to give a clearer insight into the nature of the assumptions made. In doing this, a simple mathematical property of certain bounded regions has been demonstrated which might conceivably have other uses in the theory of second moments of areas.

A new corrected form of the simple theory has been offered and its dispersion curve is better than those of the other engineering formulae for short waves. Where long waves are concerned, the new equation is inferior to Love's version but is still better than the simple theory. The method of successive approximations has been used to improve the corrected formulae (both Love's and the present one).

There is a wide literature (both theoretical and experimental) on the problems of longitudinal impact. This is almost entirely based upon the simple theory and hence makes no allowance for the dispersion effects which undoubtedly occur and

which are suggested by dimensional analysis. Davies[4] has used Love's equation in treating such problems and he points out the anomalous results which spring from so doing on account of the zero phase velocity predicted by this theory for very short waves. The new theory should overcome this difficulty and open up a way of treating longitudinal impact with better accuracy than that offered by the simple theory. The *exact* theory is not a practical engineering tool.

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A Note on Supersonic Wing Integral Equations in Unsteady Flow*

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Summary: The integral equations for the pressure distribution on an oscillating thin wing having a prescribed velocity distribution in a supersonic flow are developed.

1. Introduction

A plane wing, projecting on the region S of $z=0$, is placed in a supersonic flow of velocity U, directed along the positive x-axis, and executes a transverse motion of time dependence $\exp(i\omega t)$. If the co-ordinates are dimensionless, the linearised equation for the velocity potential is

$$\phi_{yy}+\phi_{zz}=\beta^2\phi_{xx}+2ikM\phi_x-k^2M^2\phi, \quad \phi=\phi(x,y,z) \qquad (1)$$

$$M=\frac{U}{c}, \qquad \beta=(M^2-1)^{\frac{1}{2}} \qquad (2)$$

$$k=\frac{\omega l}{U} \qquad (3)$$

where l is the characteristic length.

2. Supersonic Trailing Edge

Considering first the anti-symmetric problem where the trailing edge is everywhere supersonic, the boundary conditions on the plane $z=0+$ may be cast in the form

$$\phi_z(x,y,0)=-\alpha(x,y), \quad (x,y) \text{ in } S \qquad (4a)$$

$$\phi(x,y,0)=0, \quad (x,y) \text{ not in } S \qquad (4b)$$

*This work was done in 1948, at which time it was hoped to extend the very elegant methods available for the steady flow problem. All attempts to generalise these methods for wings of arbitrary plan form having proved nugatory, it is felt that the integral equations are worth recording.

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Equation (4*b*) is not actually valid in the wake, across which the potential is discontinuous, but suffices for the determination of the potential on the wing, in virtue of the (assumed) supersonic trailing edge. However, having the potential on the wing, the potential on the upper surface of the wake is given by

$$\phi(x, y, 0+) = \exp[-ik(x - x_1)]\,\phi(x_1, y, 0+), \quad x > x_1(y) \qquad . \qquad (5)$$

where $x_1(y)$ specifies the trailing edge.

In addition to the boundary conditions (4), it is implicit that ϕ must vanish outside the zone of action of S.

A solution to equation (1) that reduces to a prescribed potential on $z = 0+$, in consequence of Fourier's theorem, and is anti-symmetric with respect to this plane is given by

$$\phi(x, y, z) = \pm \frac{1}{4\pi^2} \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} d\mu \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \times \exp\{ i[\mu(x - \xi) + \nu(y - \eta) - \lambda |z|] \} \phi(\xi, \eta, 0+)$$

. . . . . (6)

$$\lambda = \{ \nu^2 - [(M-1)\mu + kM][(M+1)\mu + kM] \}^{\frac{1}{2}} \qquad . \quad . \quad (7)$$

where ε is a positive, real constant, placing the path of integration in the μ-plane under all singularities therein, and the branch cuts in the ν-plane are taken to infinity in the same half planes as the respective branch points from which they originate, these conditions ensuring $\mathscr{R}(\lambda) \geqslant 0$. Invoking the boundary conditions (4) and carrying out the μ and ν integrations with the aid of a table of Fourier integrals, we obtain the integral equation

$$\iint_S g(x - \xi, y - \eta)\,\phi(\xi, \eta, 0+)\,d\xi d\eta = \alpha(x, y), \quad (x, y) \text{ in } S \qquad . \qquad (8)$$

$$g(x, y) = \frac{\beta^2}{\pi R^3}[\cos(\kappa R) + (\kappa R)\sin(\kappa R)]\,e^{-i\kappa M x}, \quad x > \beta|y| \qquad . \quad . \quad (9a)$$

$$= 0, \qquad x < \beta|y| \qquad . \quad (9b)$$

$$R = +(x^2 - \beta^2 y^2)^{\frac{1}{2}} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\kappa = \frac{kM}{\beta^2} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

As written, the integral of equation (8) is improper, but this difficulty may be circumvented either by repeated integration by parts, or by adopting the well-known convention of Hadamard.

In the special case where the leading edges also are supersonic, α vanishes off the wing, and equation (8) may be inverted with the aid of the Faltung theorem to obtain the result first given by Garrick and Rubinow[1]. For a rectangular wing of effective aspect ratio greater than unity the Wiener-Hopf technique is applicable[2]. In principle, an extension of this technique, as applied by Fox to the diffraction of sound by an infinite strip[3], renders possible closed form solutions for rectangular wings of arbitrary aspect ratio, but the resulting integrals are intractable, as is also the case if a solution in Mathieu functions is attempted after introducing co-ordinates of the elliptic cylinder. (The low aspect ratio rectangular wing may be so treated, however.) A workable but involved solution for the narrow delta wing may be obtained along the lines proposed by Stewartson[4]. Useful solutions for more general plan forms would appear to be feasible only with the aid of numerical techniques.

3. Subsonic Trailing Edge

In the more general case where the trailing edge of S is partially or totally subsonic the boundary condition (4*b*) must be replaced by the requirement that the pressure, which is proportional to

$$\gamma(x,y)=\phi_x(x,y,0+)+ik\phi(x,y,0+) \quad . \quad . \quad . \quad (12)$$

must vanish for $z=0$, (x,y) not in S. Equation (6) may be extended by replacing ϕ by $-i(\mu+k)^{-1}\gamma$ in the integrand, and the resulting integral equation consequent on equation (4*a*) is

$$\iint_S h(x-\xi,y-\eta)\gamma(\xi,\eta)\,d\xi d\eta=\alpha(x,y),\ (x,y) \text{ in } S \quad . \quad . \quad (13)$$

$$h(x,y)=\int_{\beta|y|}^{x}\exp\left[i\left(\frac{k}{\beta^2}\right)(x-\xi)\right]g(\xi,y)\,d\xi,\ x>\beta|y| \quad . \quad . \quad (14a)$$

$$=0. \qquad x<\beta|y| \quad . \quad . \quad (14b)$$

The prospects of effecting a general solution to equation (13) for non-vanishing k are not encouraging.

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The Theory of Aerofoils in Unsteady Motion

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Summary: The theory of aerofoils in unsteady flow, which has made substantial progress in the last decade due largely to the ground work of Küssner and his co-workers, is presented here in a form suitable for application in aeroelastic problems, particularly those concerned with the dynamic loads on aircraft arising from gusts.

Exact expressions are given for the aerodynamic lift and moment for an oscillating aerofoil as well as for the case of arbitrary motion through disturbed air. The expressions for the latter case involve two special functions, generally referred to as Wagner and Küssner functions. Exact values of these functions are tabulated together with useful approximations. The problem of a wing-tail combination is discussed and a method of solution indicated. The bibliography at the end of the paper lists the most important publications in this field.

1. Introduction

The loads induced in aircraft arising from dynamic causes, such as from encountering disturbed air, are becoming of increasing importance as design problems. An important factor in dealing with such problems is the behaviour of aerofoils in unsteady flow, a study which has made substantial progress in recent years due in the first place to the excellent theoretical ground work done by Küssner and his co-workers about ten years ago.

It is the purpose of the present paper to summarise the theory of the two-dimensional aerofoil in incompressible unsteady flow and to present it in a form suitable for direct application to aeroelastic problems associated with subsonic aeroplanes without sweepback.

Several surveys already exist[1, 2, 3, 4] which deal with the earlier investigations into the aerodynamic forces acting on aerofoils in arbitrary motion. Hence the work done before 1936 has not been included here, particularly as the exact solution has only been obtained since then. The three-dimensional incompressible, as well as the two- and three-dimensional subsonic, problems have not yet found satisfactory solution. The supersonic problem in two dimensions[39–41] and in three dimensions for certain delta wings[42] has been solved, but may be considered of minor importance for aircraft design at present.

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The treatment of the present paper follows closely that of Küssner[5, 6] who deduced his solution from that for the oscillating aerofoil. The solution of this latter problem will only be stated here since a number of independent deductions of the relevant results may be found in Refs. 7, 8, 37 and 38.

The underlying assumptions are the same as those generally made in aerofoil theory, namely:

(*i*) the aerofoil is assumed to be thin and to approximate to a flat plate,

(*ii*) the constant forward velocity of the aerofoil is assumed to be large compared with the disturbances caused by its rotation, its lateral displacement and to the gust velocity,

(*iii*) the disturbance is allowed for by adjustment of the boundary conditions on the aerofoil and the fluid is assumed to be irrotational everywhere except in the wake,

(*iv*) the Kutta-Joukowski condition is assumed to ensure smooth flow at the trailing edge.

Some of the results, given by Küssner in 1940, had been obtained previously by himself and other investigators, but his final formula for the pressure distribution over the aerofoil and his method of deduction are different from all earlier work.

Notation

ν	frequency parameter
$2l$	chord of aerofoil
V	constant forward velocity of aerofoil
$\omega = i\nu l/V$	reduced frequency
x	chordwise co-ordinate of a point on an aerofoil such that $x = \mp 1$ corresponds to leading and trailing edges respectively (*see* Fig. 1)
W	vertical velocity
θ	$\cos^{-1}(-x)$
t	time
ϑ, σ, μ	variables of integration
Π	pressure difference
P_n, a_n	constants in equations (1) and (2) respectively
$H_{0\,(2)}, H_{1\,(2)}$	Hankel functions
C	Theodorsen function
T	$2C-1$
ρ	density

h	lateral vertical displacement of aerofoil
α	angular displacement of aerofoil
al	distance of point of rotation from centre of aerofoil
bl	distance of point about which moment is taken from centre of aerofoil
L	lift
M	moment
Π_1	instantaneous pressure difference
Π_2	transient pressure difference
s	Vt/l, non-dimensional time parameter
W_1	vertical velocity of points on the aerofoil
U_1	Wagner's function
W_2	gust velocity
W_0	amplitude of W
J_0, J_1	Bessel functions
U_2	Küssner's function
ϕ, ψ	values of θ between 0 and π (*see* equation (21))
Φ	special function describing growth of instantaneous pressure difference due to gust
k_2	$U_2+\Phi$
γ	bound vorticity
λl	distance of centre of tail section from centre of wing section
$2\xi l$	tail chord
k_1	U_1+1
ζl	distance between the centre of the wing section and a point in the wake.

2. The Harmonic Oscillating Aerofoil

Familiarity will be assumed with the concepts of bound and free vorticity and with the deduction of some of the basic results, referring to the oscillating aerofoil, which follow.

Let ν be the frequency parameter of the motion

$\omega = i\nu l/V$, the reduced frequency

V, the constant horizontal, *i.e.* forward, velocity

W, the small vertical velocity

$x=-\cos\theta$ the co-ordinate of points on the aerofoil of chord length 2, so that $\theta=0,\ \pi$, *i.e.* $x=\mp 1$ correspond to the leading and trailing edge respectively (Fig. 1).

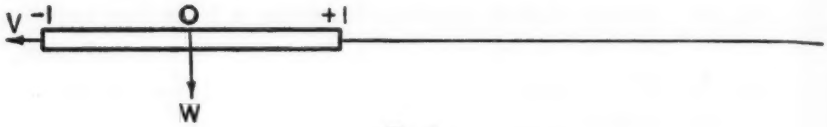

Fig. 1.

Write
$$W(\theta,t)=Ve^{i\nu t}\left[P_0+2\sum_1^{\infty}P_n\cos n\theta\right] \quad . \quad . \quad . \quad (1)$$

where the P_n are constants. The corresponding pressure difference distribution can be assumed in the form

$$\Pi(\theta,t)=\rho V^2 e^{i\nu t}\left[2a_0\cot\tfrac{1}{2}\theta+4\sum_1^{\infty}a_n(1/n)\sin n\theta\right] \quad . \quad . \quad (2)$$

where the a_n have to be determined in terms of the P_n. Using the condition of tangential flow on the aerofoil, the following general relations are found between the P_n and a_n:

$$\left.\begin{aligned} a_0&=\tfrac{1}{2}(1+T)(P_0-P_1)+P_1\\ a_n&=\tfrac{1}{2}\omega P_{n-1}-nP_n-\tfrac{1}{2}\omega P_{n+1}, \qquad n\geqslant 1\\ T&=T(-i\omega)=\frac{H_{0(2)}(-i\omega)+iH_{1(2)}(-i\omega)}{-H_{0(2)}(-i\omega)+iH_{1(2)}(-i\omega)} \end{aligned}\right\} \quad . \quad . \quad (3)$$

where $H_{0\,(2)}(-i\omega)$, $H_{1\,(2)}(-i\omega)$ are Hankel functions. It is easily verified that

$$T=2C-1,$$

where
$$C=H_{1(2)}(-i\omega)/\{H_{1(2)}(-i\omega)+iH_{0(2)}(-i\omega)\} \quad . \quad . \quad . \quad . \quad (4)$$

is Theodorsen's function[7]. The relations (3) were first stated by Küssner[2] in 1936; their detailed deduction was given by K. Jaeckel[11, 12] in 1939.

Thus it is seen from (3) that with a harmonic oscillating aerofoil, three distinct stages lead to the determination of the pressure difference distribution (2) over the aerofoil; these are:

(*a*) Calculation of the P_n as Fourier coefficients of the given $W(\theta,t)$

(*b*) computation of the a_n, using (3)

(*c*) summation of the series (2) after substitution for the a_n in terms of the P_n.

Küssner succeeded in performing some of these steps once and for all and in deducing a formula for $\Pi(\theta, t)$ which involves only a single integration over the aerofoil after substitution of the known $W(\theta, t)$. The indirect mathematical reasoning, leading to this formula, is related to the idea of the Poisson integral; it may be found in Ref. 5. The final result, determining the difference in pressure between points above and below the aerofoil for harmonic oscillations, is given by

$$\Pi(\theta,t)=\frac{\rho V}{\pi}\int_0^{\pi}\left[\frac{2\sin\theta}{\cos\theta-\cos\vartheta}+\omega\sin\vartheta\log\frac{1-\cos(\theta+\vartheta)}{1-\cos(\theta-\vartheta)}+\right.$$

$$\left.+\{1+\cos\vartheta+(1-\cos\vartheta)T(-i\omega)\}\cot\tfrac{1}{2}\theta\right]W(\vartheta,t)\,d\vartheta\ . \qquad (5)$$

Next the lift on an oscillating aerofoil, as given by Theodorsen[7] [*cf.* equation XVIII, p. 419], will be obtained, in order to demonstrate the application of (5). For this purpose let

$$W(\vartheta,t)=-\{\dot h-(a+\cos\vartheta)l\dot\alpha+V\alpha\}$$

$$=-[\{h_0-(a+\cos\vartheta)l\alpha_0\}iv+V\alpha_0]e^{ivt}\ . \qquad (6)$$

where h_0, α_0 are the amplitudes of the lateral and angular displacements h and α, $2l$ is the chord of the aerofoil and a is the distance of the rotational centre behind the centre of the aerofoil (Fig. 2).

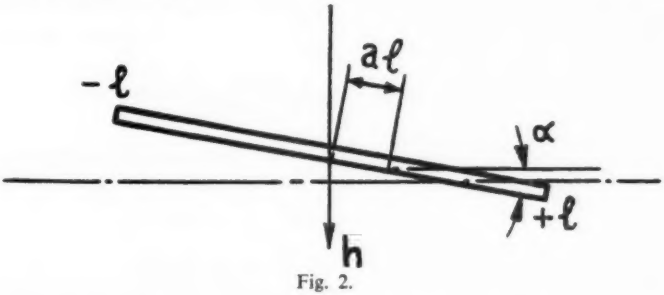

Fig. 2.

Substituting (6) in (5) and integrating with regard to ϑ we obtain

$$\Pi(\theta,t)=-\rho V[2\sin\theta\{2l\dot\alpha+(l/V)\ddot h-al^2\ddot\alpha\}-\sin\theta\cos\theta\ddot\alpha l^2/V+$$

$$+\cot\tfrac{1}{2}\theta\{(1+T)(\dot h-al\dot\alpha+V\alpha)-\tfrac{1}{2}(1-T)l\dot\alpha\}]\ . \qquad (7)$$

where, as before, dots indicate differentiation with respect to the time t.

Using (7), the lift L acting on the aerofoil at time t is given by

$$L=\int_{-l}^{l}\Pi(\theta,t)\,dx=l\int_0^{\pi}\Pi(\theta,t)\sin\theta\,d\theta\ . \qquad (8)$$

i.e. $L=-\rho V l\pi[2l\dot{\alpha}+(l/V)\{\ddot{h}-al\ddot{\alpha}\}+\{1+T\}\{\dot{h}-al\dot{\alpha}+V\alpha\}-\frac{1}{2}(1-T)l\dot{\alpha}]$

and by (4)

$$L=-\rho l^2\{V\pi\dot{\alpha}+\pi\ddot{h}-\pi a\ddot{\alpha}\}-2\pi\rho VlC\{V\alpha+h+l(\tfrac{1}{2}-a)\dot{\alpha}\} \quad . \quad . \quad (9)$$

This is Theodorsen's result after putting $V=v$ and $l=b$ and remembering that in the present case the flap angle $\beta\equiv 0$, *i.e.* that there is no aileron.

Similarly the moment about the point b is given by

$$M=-l^2\int_0^\pi \Pi(\theta,t)(b+\cos\theta)\sin\theta\,d\theta \quad . \quad . \quad . \quad . \quad (10)$$

i.e. for $b=a$:

$$M=-\rho l^2[\pi(\tfrac{1}{2}-a)Vl\alpha+\pi l^2(\tfrac{1}{8}+a^2)\ddot{\alpha}-a\pi l\ddot{h}]+$$
$$+2\rho Vl^2\pi(a+\tfrac{1}{2})C\{V\alpha+\dot{h}+l(\tfrac{1}{2}-a)\dot{\alpha}\} \quad . \quad (11)$$

(*cf.* Theodorsen[7] equation XX, p. 419)

3. The Arbitrarily Moving Aerofoil

An inspection of (5) shows that the expression for $\Pi(\theta,t)$ can be split up into the two parts

$$\Pi_1(\theta,t)=\frac{\rho V}{\pi}\int_0^\pi\left[(1+\cos\vartheta)\cot\tfrac{1}{2}\theta+\frac{2\sin\theta}{\cos\theta-\cos\vartheta}\right]W(\vartheta,t)\,d\vartheta+$$
$$+\frac{\rho l}{\pi}\int_0^\pi\log\frac{1-\cos(\theta+\vartheta)}{1-\cos(\theta-\vartheta)}\frac{\partial W(\vartheta,t)}{\partial t}\sin\vartheta\,d\vartheta \quad (12)$$

and $$\Pi_2(\theta,t)=(\rho V/\pi)T(-i\omega)\cot\tfrac{1}{2}\theta\int_0^\pi(1-\cos\vartheta)W(\vartheta,t)\,d\vartheta \quad . \quad . \quad . \quad (13)$$

where in (12) use has been made of the relation

$$\frac{l}{V}\frac{\partial W}{\partial t}=\omega W=\frac{ivl}{V}W \quad . \quad . \quad . \quad . \quad (14)$$

which holds for the harmonic oscillating aerofoil.

Küssner calls Π_1 and Π_2 the "instantaneous" and the "transient" pressure difference. Remembering that $T(-i\omega)$ originates from the presence of the wake, *i.e.* from the history of the motion, the suitability of these terms becomes obvious. Further it is clear that (12) must also hold for arbitrary motion, while (13) is only

valid for harmonic oscillations. The remainder of this section will be devoted to the task of deducing an expression for Π_2, applicable to an arbitrarily moving aerofoil.

Küssner achieves this object simply by writing down a general formula for Π_2:

$$\Pi_2(\theta, s) = \frac{\rho V}{\pi} \cot \tfrac{1}{2}\theta \int_0^{\pi} (1 - \cos \vartheta) \int_0^{s} U_1(s-\sigma) \frac{\partial W_1(\vartheta, \sigma)}{\partial \sigma} d\sigma \, d\vartheta, \quad (s>0) \qquad (15)$$

where W_1 is the vertical velocity component of points on the aerofoil, $s = Vt/l$ is a dimensionless co-ordinate along the flight path (Fig. 3) and U_1 is a function which is to play a similar role to that played by T and which has still to be determined.

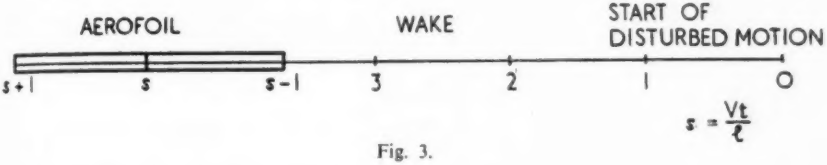

Fig. 3.

It is found by using the fact that (15) must obviously also hold true for a harmonic oscillating aerofoil, in which case it must be equal to (13) provided that $s \to \infty$. The latter condition is necessary since the function T is obtained on the assumption of an infinite wake.

Thus
$$T(-i\omega) = \omega \int_0^{\infty} e^{-\omega\sigma} U_1(\sigma) \, d\sigma \qquad (16a)$$

or by inversion of the transform,

$$U_1(s) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} T(-i\omega) \frac{e^{\omega s}}{\omega} d\omega, \quad s>0 \qquad (16b)$$

where the path of integration passes the origin on a small semi-circle on the positive side of the real axis.

In this case, the function $U_1(s)$ is to take account of the fact that any point θ of the aerofoil has the vertical velocity component $W_1(\theta, s)$ measured with respect to a zero component in steady motion. In a similar manner, by considering partial motion in the fluid and neglecting the effects of any vorticity in the fluid due solely to this motion, the transient pressure difference is obtained, caused by a "vertical gust velocity" $W_2(s)$ acting in the positive (downwards) direction. Here

$$\Pi_2(\theta, s) = \rho V \cot \tfrac{1}{2}\theta \int_0^{s+1} U_2(s-\sigma) \frac{dW_2(\sigma)}{d\sigma} d\sigma \qquad (17)$$

where the upper limit $(s+1)$ takes account of the fact that the leading edge enters the "gust" ahead of the centre of the aerofoil, from which the s co-ordinate is measured (Fig. 3). The function U_2 is again determined by reduction to the oscillating case, which now involves substitution of

$$W(\vartheta, t) = W_0 \exp(\omega s + \omega \cos\vartheta) = W_0 \exp(i\nu t - \omega x)$$

in (13) and of $$W_2(s) = W_0 e^{\omega s}$$

in (17), *i.e.* a stationary field and an oscillating aerofoil and a harmonic oscillating field are considered respectively. Equating the resulting expressions leads to

$$[J_0(-i\omega) - iJ_1(-i\omega)]\, T(-i\omega) e^{\omega s} = \omega \int_0^{s+1} e^{\omega\sigma}\, U_2(s-\sigma)\, d\sigma \quad \text{for } s > -1 \quad . \quad (18a)$$

or letting $s \to \infty$ and inverting the transform

$$U_2(s) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [J_0(-i\omega) - iJ_1(-i\omega)]\, T(-i\omega)\, e^{\omega s}\, d\omega/\omega \quad \text{for } s > -1 \quad . \quad (18b)$$

where J_0, J_1 are Bessel functions and the path of integration passes the origin on a small semi-circle on the positive side of the real axis as in (16*b*).

By the help of the functions U_1 and U_2 it is now possible to write down the general expression for the pressure difference at any point θ of the aerofoil at time $t = sl/V$:

$$\Pi(\theta, s) = \frac{\rho V}{\pi} \int_0^{\pi} \Bigg[\left\{ (1+\cos\vartheta)\cot\tfrac{1}{2}\theta + \frac{2\sin\theta}{\cos\theta - \cos\vartheta} + \right.$$

$$\left. + \sin\vartheta \log \frac{1-\cos(\theta+\vartheta)}{1-\cos(\theta-\vartheta)} \cdot \frac{\partial}{\partial s} \right\} \left\{ W_1(\vartheta, s) + W_2(s) \right\} +$$

$$+ (1-\cos\vartheta)\cot\tfrac{1}{2}\theta \int_0^{s+1} \left\{ U_1(s-\sigma) \frac{\partial W_1(\vartheta, \sigma)}{\partial \sigma} + U_2(s-\sigma) \frac{\partial W_2(\sigma)}{\partial \sigma} \right\} d\sigma \Bigg] d\vartheta \, .$$

. . . . . (19)

Here use has been made of the identity

$$(1/\pi) \int_0^{\pi} (1 - \cos\vartheta)\, d\vartheta = 1 \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and of the fact that $U_1(s) = 0$ for $s \leqslant 0$. The latter statement is obvious, since it

has been assumed that disturbances ahead of the leading edge have no influence on the motion of the aerofoil. By the same reasoning $U_2(s)=0$ for $s\leqslant -1$.

This expression (19) is Küssner's final result for the pressure difference distribution over an aerofoil in arbitrary motion. The expression involves a certain amount of vagueness since in actual fact W_2 also varies over the aerofoil, *i.e.* it is a function of s and θ. This will only affect the instantaneous part, since the transient part already involves an integration over the aerofoil and its wake. This difficulty can be overcome by a special study of the instantaneous pressure difference (12) due to a unit change in W_2 travelling across the aerofoil. Thus

$$\left.\begin{aligned} &\text{for } 0<\vartheta<\phi, \text{ let } W_2=W_0, \frac{\partial W_2}{\partial t}=0,\\ &\text{for } \phi<\vartheta<\psi, \text{ let } W_2=W_0\frac{\cos\vartheta-\cos\psi}{\cos\phi-\cos\psi}, \; l\frac{\partial W_2}{\partial t}=\frac{VW_0}{\cos\phi-\cos\psi},\\ &\text{and for } \psi<\vartheta<\pi, \text{ let } W_2=0, \frac{\partial W_2}{\partial t}=0. \end{aligned}\right\} \quad (21)$$

Substituting (21) in (12) and letting $\phi\longrightarrow\psi$ we obtain

$$\lim_{\phi\to\psi}\Pi_1(\theta,t)=\rho V W_0\left(\frac{\psi+\sin\psi}{\pi}\right)\cot\tfrac{1}{2}\theta=\rho V W_0\cot\tfrac{1}{2}\theta\,\Phi(s) \quad . \quad (22)$$

where $s=-\cos\phi$ is the co-ordinate of the sharp gust-front on the aerofoil and

$$\left.\begin{aligned} &\Phi(s)=\frac{1}{2}+\frac{1}{\pi}\sin^{-1}s+\frac{1}{\pi}(1-s^2)^{\frac{1}{2}} \text{ for } s^2<1\\ &\Phi(s)=1 \qquad\qquad\qquad\qquad \text{for } s>1 \end{aligned}\right\} \quad . \quad . \quad (23)$$

It is easily seen that in the limit as $\phi\longrightarrow\psi$ the gust velocity W_2 given by (21) corresponds to a step change W_0 at the point $-\cos\phi$ and that hence, by (22), the function $\Phi(s)$ describes the building up of instantaneous pressure as a unit change in the gust velocity traverses the aerofoil.

In the case of an arbitrary gust $W_2(s)$, the instantaneous pressure difference due to it can now be expressed, using the function Φ, as

$$\Pi_1(\theta,s)=\rho V\cot\tfrac{1}{2}\theta\int_0^{s+1}\Phi(s-\sigma)\frac{\partial W_2(\sigma)}{\partial\sigma}\,d\sigma \quad . \quad . \quad (24)$$

Using (24) and (20) the formula (19) can now be rewritten

$$\Pi(\theta,s)=\frac{\rho V}{\pi}\int_0^{\pi}\Big[\Big\{(1+\cos\vartheta)\cot\tfrac{1}{2}\theta+ \\ +\frac{2\sin\theta}{\cos\theta-\cos\vartheta}+\sin\vartheta\log\frac{1-\cos(\theta+\vartheta)}{1-\cos(\theta-\vartheta)}\cdot\frac{\partial}{\partial s}\Big\}W_1(\vartheta,s)+ \\ +(1-\cos\vartheta)\cot\tfrac{1}{2}\vartheta\int_0^{s+1}\Big\{U_1(s-\sigma)\frac{\partial W_1(\vartheta,\sigma)}{\partial\sigma}+k_2(s-\sigma)\frac{\partial W_2(\sigma)}{\partial\sigma}\Big\}\,d\sigma\Big]\,d\vartheta \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where

$$k_2(s)=U_2(s)+\Phi(s). \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Finally note that in (25) the integral with respect to σ may also take account of discontinuous changes in the velocity functions.

4. Arbitrary Motion of a Wing-tail Combination

Since

$$\Pi(\theta,s)=\rho V\gamma(\theta,s) \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where $\gamma(\theta,s)$ is the bound vorticity distribution over the aerofoil, it is seen from (25) that in the case of arbitrary motion

$$\gamma(\theta,s)=\frac{1}{\pi}\int_0^{\pi}\Big[\Big\{(1+\cos\vartheta)\cot\tfrac{1}{2}\theta+ \\ +\frac{2\sin\theta}{\cos\theta-\cos\vartheta}+\sin\vartheta\log\frac{1-\cos(\theta+\vartheta)}{1-\cos(\theta-\vartheta)}\cdot\frac{\partial}{\partial s}\Big\}W_1(\vartheta,s)+ \\ +(1-\cos\vartheta)\cot\tfrac{1}{2}\vartheta\int_0^{s+1}\Big\{U_1(s-\sigma)\frac{\partial W_1(\vartheta,\sigma)}{\partial\sigma}+k_2(s-\sigma)\frac{\partial W_2(\sigma)}{\partial\sigma}\Big\}\,d\sigma\Big]\,d\vartheta \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Thus the bound vorticity distribution for any particular motion of the aerofoil is easily found.

When considering a wing-tail combination, in addition to the forces due to the motion of the tail section and the gust, there will be effects resulting from the presence of the wing. The former forces will be given again by an expression of the form (25), the only difference being that $(s-\sigma)$ must be replaced by $(s-\sigma)/\xi$, where $2\xi l$ is the chord of the tail-section. Further account must be taken of the fact that the gust velocity function $W_2(\sigma)$ will not act before σ attains the value $\lambda-\xi$ where λl is equal to the distance between the centres of the wing and the tail sections (Fig. 4).

The general problem for the effects of the wing and its wake on the aerodynamic forces on the tail is very much more involved. It is customary to assume that the chords of the wing and tail sections are parts of the same straight line which contains their wakes. Although this assumption is not essential it simplifies considerably the expression to be found later for the velocity induced by the vortex system of the wing on the tailplane. Once this induced velocity has been determined it can be added to the gust velocity $W_2(\sigma)$ in the expression discussed in the preceding paragraph. Another assumption, which is not in agreement with actual facts,

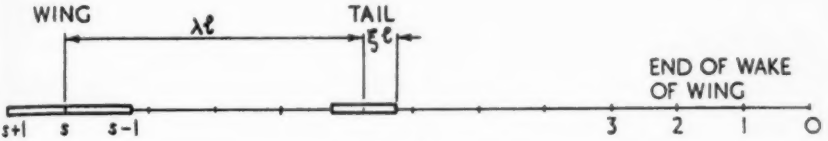

Fig. 4.

concerns the effective incidences of the wing and tail, which have been assumed equal. This should not influence the final result to any great extent, since only changes from the equilibrium condition, *i.e.* from the condition of steady motion, are important.

In order to determine the induced velocity on the tail it is necessary to determine the free vorticity distribution over the wing section and its wake. It can be found from the bound vorticity in the following manner, since

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial s}\frac{ds}{dt} = \frac{V}{l}\frac{\partial \gamma(\theta, s)}{\partial s} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

is the variation of the bound vorticity γ at time $t = ls/V$ at the point $(-l\cos\theta)$. The free vorticity at any point $(-l\cos\theta_1)$ on the aerofoil may be defined as being equal in magnitude and opposite in sign to the accumulation of the bound vorticity variations generated between the point under consideration and the leading edge. In this connection attention must be given to the fact that these variations move along the aerofoil with velocity V so that the variations constituting the free vorticity at $(-l\cos\theta_1)$ at time ls/V originated at the earlier time $(l/V\{s-(\cos\theta-\cos\theta_1)\})$ at the point $(-l\cos\theta)$. Remembering that $\theta = 0$ corresponds to the leading edge, we have for the free vorticity

$$-\int_0^{\theta_1} \frac{V}{l}\frac{\partial \gamma}{\partial s}[\theta, s-(\cos\theta-\cos\theta_1)](-l\sin\theta)\,d\theta$$

$$= V\int_0^{\theta_1} \frac{\partial \gamma}{\partial s}[\theta, s-(\cos\theta-\cos\theta_1)]\sin\theta\,d\theta \quad . \quad . \quad . \quad . \quad . \quad (30)$$

In particular, the free vorticity at the trailing edge is given by

$$V \int_0^{\pi} \frac{\partial \gamma}{\partial s}(\theta, s - \cos\theta - 1) \sin\theta \, d\theta \quad . \quad . \quad . \quad . \quad (31)$$

and hence at any point ζl of the wake by:

$$V \int_0^{\pi} \frac{\partial \gamma}{\partial s}(\theta, s - \cos\theta - \zeta) \sin\theta \, d\theta \quad . \quad . \quad . \quad . \quad (32)$$

where ζl is the distance between the centre of the wing section and the point in the wake under consideration.

Using these expressions for the free vorticity the velocity induced by the wing vortex system on the tail can be written as

$$-\frac{V}{2\pi} \int_1^{\lambda} \frac{d\mu}{\mu - \zeta} \int_0^{\pi} \frac{\partial \gamma}{\partial s}(\theta, s - \cos\theta - \zeta) \sin\theta \, d\theta -$$

$$-\frac{V}{2\pi} \int_0^{\pi} \frac{\sin\theta_1 \, d\theta_1}{\cos\theta_1 - \zeta} \int_0^{\theta_1} \frac{\partial \gamma}{\partial s}[\theta, s - (\cos\theta - \cos\theta_1)] \sin\theta \, d\theta + \frac{1}{2\pi} \int_0^{\pi} \frac{\gamma(\theta, s)}{\cos\theta - \zeta} \sin\theta \, d\theta$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where $\lambda - \xi \leqslant \zeta \leqslant \lambda + \xi$ is the range of the tail section.

As already mentioned, this induced velocity can be added to the gust velocity term in the expression (25) after modification for the tail section.

5. Remarks on the Special Functions

It is seen from the final expression (25) for the pressure difference distribution that the functions U_1, U_2 and k_2 are as fundamental to the problem of an aerofoil in arbitrary motion as is the function T (*see* (3)) to the problem of an oscillating aerofoil. These functions have been investigated by different authors who also spent considerable time on their tabulation. The formulae (16*b*) and (18*b*) had been deduced previously by Küssner[2] and I. E. Garrick[13, 14]. In their present form they are of little value for the determination of the actual values of the functions. Detailed analytical investigations of U_1, U_2 and k_2 can be found in Ref. 6 (Section VI) and in a paper by L. Schwarz[16].

In analogy with (26), *i.e.*

$$k_2(s) = U_2(s) + \Phi(s) \qquad s > -1 \quad . \quad . \quad . \quad (34)$$

it is convenient to introduce a function $k_1(s)$ defined by

$$k_1(s)=U_1(s)+1 \qquad \text{for } s>0 \quad . \quad . \quad . \quad (35)$$

The analogy will be still clearer when reference is made to (23), for it is seen then that

$$k_2(s)=U_2(s)+1,\ k_1(s)=U_1(s)+1 \text{ for } s>1. \quad . \quad . \quad (36)$$

The function $k_1(s)$ was first calculated by H. Wagner[15]. Küssner[2] deduced the function k_2 and tabulated it together with k_1 in 1936. However, he made a mistake in the determination of k_2 which was first pointed out by von Kármán and Sears[17] in 1938. In 1940 Küssner gave corrected values of k_1 and k_2 which are reproduced in Tables I and II towards the end of this paper. While these tables are sufficient for use with any approximate integration formula, there are also approximate expressions for U_1, U_2 and k_2 which have been given by various authors. Some of these will now be given here and conclusions with regard to their accuracy can be drawn by comparison of the relevant columns of the Tables I and II in which some of these expressions have been tabulated.

In chronological order the approximations are:

I. E. Garrick (1938, Ref. 13): $$k_1(s)=2-\frac{4}{4+s} \quad . \quad . \quad . \quad . \quad (37)$$

Garrick's approximation to $k_2(s)$ was incorrect due to the use he made of faulty results drawn from Küssner's paper[2].

I. E. Garrick (1939, Ref. 14):

$$k_2(s)=\left\{\begin{array}{ll} (2/\pi)\,3\cos^{-1}(-s)-4\left(\frac{s+5}{s+3}\right)^{1/2}\sin^{-1}\left(\frac{(s+1)(s+3)}{8}\right)^{1/2}, & |s|<1 \\ 6-4\left(\frac{s+5}{s+3}\right)^{1/2} & ,s<1 \end{array}\right\} \quad (38)$$

This expression is the result of substituting (37) into the following formula, relating $U_1(s)$ and $U_2(s)$:

$$U_2(s)=\frac{1}{\pi}\int_{-1}^{+1}U_1(s-x)\left(\frac{1+x}{1-x}\right)^{1/2}dx. \quad . \quad . \quad . \quad (39)$$

R. T. Jones (1939, Ref. 18):

$$k_1(s)=2-0.33\,e^{-0.0455s}-0.67\,e^{-0.3s} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

R. T. Jones (1940, Ref. 19):

$$k_2(s)=2-0.472\,e^{-0.058(s+1)}-1.026\,e^{-0.364(s+1)}-0.342\,e^{-2.42(s+1)} \quad . \quad (41)$$

W. R. Sears and B. O. Sparks (1941, Ref. 30):

$$k_2(s)=2-1.0\,e^{-0.130(s+1)}-1.0\,e^{-(s+1)} \quad . \quad . \quad . \quad . \quad . \quad (42)$$

W. Prichard Jones (1945, Ref. 20):

$$k_1(s)=2-0.33\,e^{-0.041s}-0.67\,e^{-0.32s}. \quad . \quad . \quad . \quad . \quad . \quad (43)$$

In Ref. 20 the author also confirms Küssner's computations of $k_2(s)$ and reprints his table. It should also be noted here that some authors use $\pi k_1(s)$ and $\pi k_2(s)$ instead of $k_1(s)$ and $k_2(s)$; while W. P. Jones denotes the new functions again by $k_1(s)$ and $k_2(s)$, R. T. Jones uses the symbols $C_{L\alpha}(s)$ and $C_{L\dot{\alpha}}(s)$. Further, in some places the argument of the functions $U_2(s)$ or $k_2(s)$ starts from $s=0$ instead of $s=-1$.

Finally certain physical interpretations of the functions $U_1(s)$ and $U_2(s)$ will be given, which will be of particular value in connection with the various applications of the theory in engineering problems, since that is the way in which some authors introduce these functions. Thus $U_1(s)$ describes the growth of the transient pressure difference following a sudden unit change in the vertical velocity component of the aerofoil itself. The function $U_2(s)$ takes account of the growth of the transient pressure difference subsequent to the entrance of an aerofoil into a sharp edged unit gust, while the function $k_2(s)$ represents the growth of the total pressure difference following a sudden unit change in the gust velocity. [For $\Phi(s)$ *cf.* remarks following (23).] On the basis of these interpretations the formulae (15) and (17) can be deduced by some physical reasoning similar to that used before for obtaining the expression (24) for the instantaneous pressure difference subsequent to a sharp gust.

6. Expressions for Aerodynamic Forces Used in Applications

Note: In the whole of this Section ′ denotes differentiation with respect to s.

The final formula (25) gives only the pressure difference distribution over an aerofoil in arbitrary motion. One further integration over the aerofoil is necessary, therefore, in order to obtain expressions for the lift and aerodynamic moment acting on the aerofoil [*see* (8) and (10)]. Thus, assuming that the aerofoil is rigid and is subject to vertical and angular displacements h and α, where the latter occurs about an axis inside the aerofoil, and to changes in the vertical velocity component of the fluid, we can write for $W_1(\vartheta, s)$:

$$W_1(\vartheta,s)=-\left[\frac{dh}{dt}+V\alpha-(a+\cos\vartheta)\,l\frac{d\alpha}{dt}\right]=-\left[\dot{h}+V\alpha-(a+\cos\vartheta)\,l\alpha\right]$$

$$=-\dot{s}\left\{\frac{dh}{ds}+l\alpha-(a+\cos\vartheta)\,l\frac{d\alpha}{ds}\right\}=-\frac{V}{l}\left\{h'+l\alpha-(a+\cos\vartheta)\,l\alpha'\right\}$$

$$. \quad . \quad . \quad . \quad . \quad (44)$$

(*see* Figs. 2, 3). Substituting (44) and an arbitrary $W_2(s)$ in (25) and integrating with regard to ϑ gives

$$\Pi(\theta, s)=\rho V\left[\cot\tfrac{1}{2}\theta\left\{A-\tfrac{1}{2}B+C\right\}+2\sin\theta\left\{A'+B\right\}-\sin\theta\cos\theta\,B'\right] \quad (45)$$

where
$$\left.\begin{aligned} A(s)&=-(V/l)\left\{h'+l\alpha-a\,l\alpha'\right\},\quad B(s)=-V\alpha' \\ C(s)&=\int_0^{s+1}\left[-U_1(s-\sigma)\left\{A'+\tfrac{1}{2}B'\right\}+k_2(s-\sigma)W_2'\right]d\sigma. \end{aligned}\right\} \quad (46)$$

Substituting (45) in (8) and (10) and integrating with respect to θ over the aerofoil, the exact expressions are obtained finally for the variations in lift L and moment M from the equilibrium position of steady motion.

$$\begin{aligned} dL&=\rho\pi lV\left[A+A'+\tfrac{1}{2}B+C\right] \\ &=-\rho\pi V^2\left[h'+h''+l\alpha+l\alpha'\left\{\tfrac{3}{2}-a\right\}-a\,l\alpha''+\int_0^{s+1}U_1(s-\sigma)\left\{h''+l\alpha'+l\alpha''\left(\tfrac{1}{2}-a\right)\right\}d\sigma\right]+ \\ &\qquad+\rho\pi lV\int_0^{s+1}k_2(s-\sigma)W_2'\,d\sigma \qquad . \quad . \quad . \quad . \quad (47) \end{aligned}$$

$$\begin{aligned} dM&=\rho\pi l^2V\left[-\left\{b+\tfrac{1}{2}\right\}\left\{A-\tfrac{1}{2}B+C\right\}-b\left\{A'+B\right\}+B'/8\right] \\ &=-\rho\pi lV^2\left[-\left\{b+\tfrac{1}{2}\right\}\left\{h'+l\alpha-l\alpha'(a+\tfrac{1}{2})+\int_0^{s+1}U_1(s-\sigma)\left\{h''+l\alpha'+(\tfrac{1}{2}-a)l\alpha'\right\}d\sigma\right\}-\right. \\ &\quad\left.-b\left\{h''+2\,l\alpha'-a\,l\alpha''\right\}+l\alpha''/8\right]-\rho\pi l^2V\left\{b+\tfrac{1}{2}\right\}\int_0^{s+1}k_2(s-\sigma)W_2'\,d\sigma \qquad . \quad (48) \end{aligned}$$

or using (35), *i.e.*

$$U_1(s)=k_1(s)-1,\quad s>0,$$

$$\begin{aligned} dL&=-\rho\pi V^2\left[h''+l\alpha'-a\,l\alpha''+\int_0^{s+1}k_1(s-\sigma)\left\{h''+l\alpha'+l\alpha''\left(\tfrac{1}{2}-a\right)\right\}d\sigma\right]+ \\ &\qquad+\rho\pi lV\int_0^{s+1}k_2(s-\sigma)W_2'\,d\sigma. \quad . \quad (49) \end{aligned}$$

$$\begin{aligned} dM&=-\rho\pi l^2V^2\left[-(b/l)h''+\alpha'\left(\tfrac{1}{2}-b\right)+\alpha''\left(\tfrac{1}{8}+ab\right)-\right. \\ &\qquad\left.-(b+\tfrac{1}{2})\int_0^{s+1}k_1(s-\sigma)\left\{h''/l+\alpha'+\alpha''\left(\tfrac{1}{2}-a\right)\right\}d\sigma\right] \\ &\qquad=\rho\pi l^2V\left\{b+\tfrac{1}{2}\right\}\int_0^{s+1}k_2(s-\sigma)W_2'\,d\sigma \qquad . \quad . \quad . \quad (50) \end{aligned}$$

Note that as before *al* is the centre of pitch of the aerofoil and *bl* is the point about which the moment is taken.

In the case of a wing-tailplane combination similar expressions will be obtained, the main difference arising out of the term (33), which is to be added to $W_2(s/\xi)$, the time lag, due to the position of the tailplane behind the wing section, and the modification to be made to $W_1(\vartheta, s/\xi)$ as given by (44). In actual fact $W_1(\vartheta, s/\xi)$ becomes in this case

$$W_1\left(\vartheta, \frac{s}{\xi}\right) = -\frac{V}{l}\{h' + l\alpha + (\lambda - a - \xi\cos\theta)l\alpha\} \quad . \quad . \quad (51)$$

where, as in Section 4, the chord of the tailplane is given by $2\xi l$.

In many applications the co-ordinate s along the path of the aerofoil is measured from the leading edge instead of from the centre of the aerofoil so that the upper limits of the integrals entering into the expressions for the lift and moment become s instead of $(s+1)$.

It is often more convenient to present the formulae (49) and (50) in a different manner, the principle of which is to divide the lift force into localised components and to state with each of these their point of application so that the total lift and the moment about any point *bl* of the aerofoil can be written down. By this method of representation the formulae given previously take the form (positive forces acting downwards):

$$\left.\begin{array}{ll}
\text{(a) } \textit{Inertia forces} & \\
\quad -\rho\pi V^2\{h'' - a l\alpha''\} & \text{at } \tfrac{1}{2}\text{-chord point} \\
\quad -\rho\pi l V^2 \alpha' & \text{at } \tfrac{3}{4}\text{-chord point} \\
\text{(b) } \textit{Wagner force} & \\
\quad -\rho\pi l V^2 \displaystyle\int_0^s k_1(s-\sigma)\{h''/l + \alpha' + \alpha''(\tfrac{1}{2} - a)\}\, d\sigma & \text{at } \tfrac{1}{4}\text{-chord point} \\
\text{(c) } \textit{Küssner force} & \\
\quad \rho\pi l V \displaystyle\int_0^s k_2(s-\sigma) W_2'\, d\sigma & \text{at } \tfrac{1}{4}\text{-chord point} \\
\text{(d) } \textit{Pure couple} & \\
\quad -\rho\pi l^2 V^2 \alpha''/8\,. &
\end{array}\right\} \quad . \quad (52)$$

This classification of the various lift components is self-explanatory, the first heading being due to H. Glauert[22], who gave it a physical interpretation.

Naturally the representation (52) is not unique, but the one given here is that mostly occurring in applications. Further it was pointed out at the end of Section 4 that the expressions for the Wagner- and Küssner-forces are such as to take

account of discontinuous disturbances. In applications, the discontinuities often occur at the beginning of the disturbances (sharp-edged gusts) for which reason the terms just given for the Wagner- and Küssner-forces are augmented respectively by

$$-\rho\pi lV^2 k_1(s)\{(1/l)h'(0)+\alpha(0)+\alpha'(0)\}, \quad \rho\pi lV k_2(s) W_2(0)$$

where $h'(0)$, $\alpha(0)$, $\alpha'(0)$ and $W_2(0)$ refer to initial discontinuous changes from the undisturbed state.

In References 21, 24 and 28 the aerodynamic forces have been presented in a similar manner as in (52). It should be noted that in the first of these references the centre of rotation is the quarter-chord point ($a=-\frac{1}{2}$). In the second reference[24] the centre of rotation is at the three-quarter chord point ($a=\frac{1}{2}$). In the last paper[28] the inertia term due to the angular acceleration is absent.

Finally, mention should be made of two papers by von Kármán and Sears[17] and by Sears alone because they have given an independent deduction of Küssner's results and pointed out a mistake made by Küssner[2] in 1936. These authors, defining aerodynamic lift and moment, respectively, as the rate of change of the total momentum and of the total moment of momentum of the aerofoil and wake, based their treatment on circulation theory which Küssner avoided intentionally in his last paper[6]. The second paper by Sears is of special interest since it opens up new methods for the study of the special functions U_1 and U_2.

7. Three-dimensional Effects

For the sake of completeness a few remarks will now be made about three-dimensional effects, taken into account by various writers. The general literature on three-dimensional aerofoil theory is larger than that on two-dimensional theory, but as mentioned in the introduction, it is still far from complete. For this reason all investigators of gust load problems have restricted themselves to the use of the results of the two-dimensional theory, but it is customary to make allowance for finite wings by replacing the factor 2π in the expressions for the aerodynamic forces and moments by the actual slope of the relevant lift coefficient curve in steady flight. However, this procedure may lead to errors with wings of low aspect ratio. There is also another factor, mostly taken into account by consideration of three-dimensional effects, which is the downwash behind the wing and its effects on the tailplane. The procedure is similar to the one adopted in Section 4, the main difference being that account is taken of an inherent instability in the vortex sheet forming the wake.

Among recent investigators of three-dimensional aerofoil theory the most important are H. G. Küssner[31], L. Schwarz[32], W. Prichard Jones[20] and E. Reissner[33–35], most of whom consider only the case of a harmonic oscillating wing. It should again be possible to deduce the arbitrary case from the oscillating one, once a satisfactory solution has been found for the other.

TABLE I

WAGNER'S FUNCTION

s	$U_1(s)$ (Ref. 6)	$k_1(s)$ (Ref. 6)	*Approximations to* $k_1(s)$		
			Equation (37)	*Equation* (40)	*Equation* (43)
0.0	0.000 00	1.000 00	1.000 00	1.000 00	1.000 00
0.1	0.024 39	1.024 39	1.024 40	1.021 30	1.022 45
0.2	0.047 64	1.047 64	1.047 62	1.042 01	1.044 24
0.3	0.069 82	1.069 82	1.069 76	1.062 14	1.065 36
0.4	0.091 03	1.091 03	1.090 90	1.081 71	1.085 87
0.5	0.111 33	1.111 33	1.111 12	1.100 74	1.105 77
0.6	0.130 78	1.130 78	1.130 44	1.119 26	1.125 06
0.7	0.149 45	1.149 45	1.148 94	1.137 26	1.143 80
0.8	0.167 38	1.167 38	1.166 66	1.154 76	1.161 98
0.9	0.184 62	1.184 62	1.183 68	1.171 78	1.179 62
1.0	0.201 21	1.201 21	1.200 00	1.188 33	1.196 74
1.1	0.217 19	1.217 19	1.215 68	1.204 43	1.213 35
1.2	0.232 60	1.232 60	1.230 78	1.220 09	1.229 48
1.3	0.247 47	1.247 47	1.245 28	1.235 33	1.245 14
1.4	0.261 82	1.261 82	1.259 26	1.250 14	1.260 35
1.5	0.275 69	1.275 69	1.272 72	1.264 56	1.275 10
1.6	0.289 10	1.289 10	1.285 72	1.278 59	1.289 42
1.7	0.302 07	1.302 07	1.298 24	1.292 23	1.303 34
1.8	0.314 63	1.314 63	1.310 34	1.305 51	1.316 85
1.9	0.326 79	1.326 79	1.322 04	1.318 42	1.329 96
2.0	0.338 58	1.338 58	1.333 34	1.331 00	1.342 70
2.1	0.350 01	1.350 01	1.344 26	1.343 23	1.355 06
2.2	0.361 09	1.361 09	1.354 84	1.355 14	1.367 08
2.3	0.371 85	1.371 85	1.365 08	1.366 73	1.378 75
2.4	0.382 30	1.382 30	1.375 00	1.378 02	1.390 08
2.5	0.392 45	1.392 45	1.384 62	1.388 99	1.401 10

TABLE I (*continued*)

s	$U_1(s)$ (Ref. 6)	$k_1(s)$ (Ref. 6)	*Approximations to* $k_1(s)$		
			Equation (37)	*Equation* (40)	*Equation* (43)
3.0	0.4391	1.4391	1.428 58	1.439 70	1.451 65
3.5	0.4799	1.4799	1.466 66	1.484 12	1.495 50
4.0	0.5159	1.5159	1.500 00	1.523 11	1.533 63
4.5	0.5479	1.5479	1.529 42	1.557 41	1.566 86
5.0	0.5764	1.5764	1.555 56	1.587 65	1.595 90
5.5	0.6020	1.6020	1.578 94	1.614 39	1.621 35
6.0	0.6251	1.6251	1.600 00	1.638 09	1.643 74
6.5	0.6460	1.6460	1.619 04	1.659 17	1.663 50
7.0	0.6650	1.6650	1.636 36	1.677 96	1.681 00
7.5	0.6824	1.6824	1.652 18	1.694 79	1.696 58
8.0	0.6983	1.6983	1.666 66	1.709 91	1.710 49
8.5	0.7128	1.7128	1.680 00	1.723 53	1.722 96
9.0	0.7263	1.7263	1.692 30	1.735 86	1.734 22
9.5	0.7386	1.7386	1.703 70	1.747 06	1.744 41
10	0.7501	1.7501	1.714 28	1.757 27	1.753 69
11	0.7706	1.7706	1.733 34	1.755 23	1.769 96
12	0.7883	1.7883	1.750 00	1.790 53	1.783 84
15	0.8296	1.8296	1.789 48	1.825 79	1.816 08
20	0.8733	1.8733	1.833 34	1.865 51	1.853 55
25	0.9003	1.9003	1.862 06	1.893 83	1.881 38
30	0.9183	1.9183	1.882 36	1.915 64	1.903 50
40	0.9405	1.9405	1.909 10	1.946 53	1.935 99
50	0.9535	1.9535	1.925 92	1.966 08	1.957 52
100	0.978 12	1.978 12	1.961 538	1.996 513	1.994 531
500	0.995 903	1.995 903	1.992 064	2.000 000	2.000 000
1000	0.997 973	1.997 973	1.996 016	2.000 000	2.000 000
∞	1.000 000	2.000 000	2.000 000	2.000 000	2.000 000

TABLE II

KÜSSNER'S FUNCTION

s	$U_2(s)$ (Ref. 6)	$k_2(s)$ (Ref. 6)	$\Phi(s)$	*Approximations to* $k_2(s)$		
				Equation (38)	*Equation* (41)	*Equation* (42)
−1.0	0.000 00	0.000 00	0.000 00	0.000 00	0.160 00	0.000 00
−0.9	0.000 05	0.282 36	0.282 31	0.282 35	0.272 92	0.108 08
−0.8	0.000 27	0.396 09	0.395 82	0.396 10	0.368 70	0.206 93
−0.7	0.000 73	0.481 24	0.480 51	0.481 27	0.450 81	0.297 43
−0.6	0.001 51	0.551 32	0.549 81	0.551 33	0.521 93	0.380 35
−0.5	0.002 63	0.611 63	0.609 00	0.611 66	0.584 24	0.456 40
−0.4	0.004 15	0.664 90	0.660 75	0.664 89	0.639 38	0.526 23
−0.3	0.006 12	0.712 78	0.706 66	0.712 73	0.688 71	0.590 39
−0.2	0.008 56	0.756 35	0.747 79	0.756 32	0.733 26	0.649 44
−0.1	0.011 53	0.796 36	0.784 83	0.796 36	0.773 88	0.703 84
0.0	0.015 08	0.833 39	0.818 31	0.833 34	0.811 23	0.754 02
+0.1	0.019 25	0.867 85	0.848 60	0.867 78	0.845 83	0.800 38
0.2	0.024 10	0.900 07	0.875 97	0.899 99	0.878 09	0.843 25
0.3	0.029 70	0.930 33	0.900 63	0.930 24	0.908 36	0.882 96
0.4	0.036 11	0.958 84	0.922 73	0.958 69	0.936 90	0.919 80
0.5	0.043 45	0.985 78	0.942 33	0.985 61	0.963 94	0.954 04
0.6	0.051 83	1.011 31	0.959 48	1.011 04	0.989 63	0.985 89
0.7	0.061 42	1.035 55	0.974 13	1.035 23	1.014 14	1.015 60
0.8	0.072 47	1.058 62	0.986 15	1.058 25	1.037 56	1.043 34
0.9	0.085 44	1.080 62	0.995 18	1.080 09	1.060 02	1.069 29
1.0	0.101 62	1.101 62	1.000 00	1.100 98	1.081 57	1.093 61
1.5	0.194 2	1.194 2	$\Phi(s)=1$ for $s>1$	1.192 60	1.177 91	1.195 39
2.0	0.270 3	1.270 3		1.267 12	1.258 88	1.273 15
2.5	0.334 4	1.334 4		1.329 00	1.327 67	1.335 35
3.0	0.389 1	1.389 1		1.381 20	1.386 48	1.387 16

TABLE II (*continued*)

s	$U^2(s)$ (Ref. 6)	$k_2(s)$ (Ref. 6)	$\Phi(s)$	*Approximations to* $k_2(s)$		
				Equation (38)	*Equation* (41)	*Equation* (42)
3.5	0.436 4	1.436 4		1.425 84	1.437 00	1.431 78
4.0	0.477 7	1.477 7		1.464 44	1.480 58	1.471 21
4.5	0.514 0	1.514 0		1.498 16	1.518 34	1.506 72
5.0	0.546 3	1.546 3		1.527 88	1.551 20	1.539 11
5.5	0.575 0	1.575 0		1.554 24	1.579 96	1.568 94
6.0	0.600 8	1.600 8		1.577 84	1.605 23	1.596 57
6.5	0.624 1	1.624 1		1.599 04	1.627 58	1.622 26
7.0	0.645 1	1.645 1		1.618 20	1.647 45	1.646 21
7.5	0.664 2	1.664 2		1.635 64	1.665 21	1.668 59
8.0	0.681 7	1.681 7		1.651 52	1.681 19	1.689 51
8.5	0.697 6	1.697 6	$\Phi(s)=1$	1.666 12	1.695 64	1.709 10
9.0	0.712 3	1.712 3		1.679 48	1.708 80	1.727 42
9.5	0.725 8	1.725 8	for	1.691 88	1.720 83	1.744 59
10	0.738 2	1.738 2		1.703 32	1.731 90	1.760 67
11	0.760 3	1.760 3		1.723 80	1.751 66	1.789 85
12	0.779 5	1.779 5	$s>1$	1.741 68	1.768 89	1.815 48
15	0.823 5	1.823 5		1.783 64	1.810 37	1.875 07
20	0.869 8	1.869 8		1.829 72	1.859 88	1.934 78
25	0.898 0	1.898 0		1.859 60	1.895 44	1.965 95
30	0.916 7	1.916 7		1.880 56	1.921 81	1.982 23
40	0.939 7	1.939 7		1.908 04	1.956 23	1.995 16
50	0.953 0	1.953 0		1.925 24	1.975 49	1.998 68
100	0.978 00	1.978 00		1.961 36	1.998 65	2.000 00
500	0.995 898	1.995 898		1.992 04	2.000 00	2.000 00
1000	0.997 972	1.997 972		1.996 01	2.000 00	2.000 00
∞	1.000 000	2.000 000		2.000 00	2.000 00	2.000 00

8. Conclusion

The aerodynamic theory, summarised in this paper is exact within the basic assumptions of two-dimensional incompressible aerofoil theory. Application of its results to the problem of gust loads on resilient aircraft has been given in a number of papers dealing with rigid and elastic aeroplanes. Because of the three-dimensional features of actual aircraft any such application can produce only approximate solutions. On the other hand it is known that the dynamic representation of structures of the complexity of aircraft is possible only in a very approximate manner. In view of this it appears that the results deduced here will be sufficiently accurate when dealing with the problem of gust loads on conventional low speed aircraft. This state of affairs will only be changed when some of the basic assumptions are no longer satisfied. For example, an investigation of swept-back wings cannot use these results since, quite apart from difficulties of structural representation, the flow around such wings cannot be approximated closely by two-dimensional conditions.

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